QUANTUM TIME

By

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Abstract

In quantum mechanics, time plays a role unlike any other observable. We find that measuring hether an event happened, and measuring hen an event happened are fundamentally different – the t o measurements do not correspond to compatible observables and interfere ith each other. We also propose a basic limitation on measurements of the arrival time of a free particle given by $1/\bar{E}_k$ here $\bar{E}_k$ is the particle's kinetic energy. The temporal order of events is also an ambiguous concept in quantum mechanics. It is not al ays possible to determine hether one event lies in the future or past of another event. One cannot measure hether one particle arrives to a particular location before or after another particle if they arrive ithin a time of $1/\bar{E}$ of each other, here $\bar{E}$ is the total kinetic energy of the t o particles. These ne inaccuracy limitations are dynamical in nature, and fundamentally different from the Heisenberg uncertainty relations. They refer to individual measurements of a single quantity. It is hoped that by understanding the role of time in quantum mechanics, e may gain ne insight into the role of time in a quantum theory of gravity.

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Dedication

In loving memory of my father, Peter Oppenheim (1942-1998) - my first physics teacher, ho encouraged my curiosity, patiently ans ered my questions, and patiently asked his o n. He ould have loved to flip through this thing, and I had al ays imagined giving him a copy.

Acknowledgements

I ould like to thank my Ph.D. advisor, Bill Unruh, not only for teaching me a great deal of physics, but also for teaching me ho to attack problems and discover interesting questions. For allo ing me to ander off in other directions and subtle guidance.

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Much of this ork is contained in [1] - [7], and as done in collaboration ith Yakir Aharanov, Benni Reznik, Sandu Popescu, and Bill Unruh. Thanks to my collaborators for their help, for many interesting discussions, insights and paradoxes.

Throughout my ork, I have had the benefit of discussions ith a number of researchers. Carlo Rovelli, Gonzalo Muga, Rick Leavens, and Bob Wald have been particularly helpful.

Thanks also to my committee members, Kristin Schleich, Mark Halpern, Gordon Semenoff, and Ian Affleck for their suggestions, and to fello graduate students Lori Paniak, Adam Monahan, Richard Szabo, Todd Fugleburg, Martin Dube, Sebastian Jaimungal and others for their help.

Special thanks to the amazing Victoria Scott for dra ing the cartoons for this thesis, as ell as for everything else.

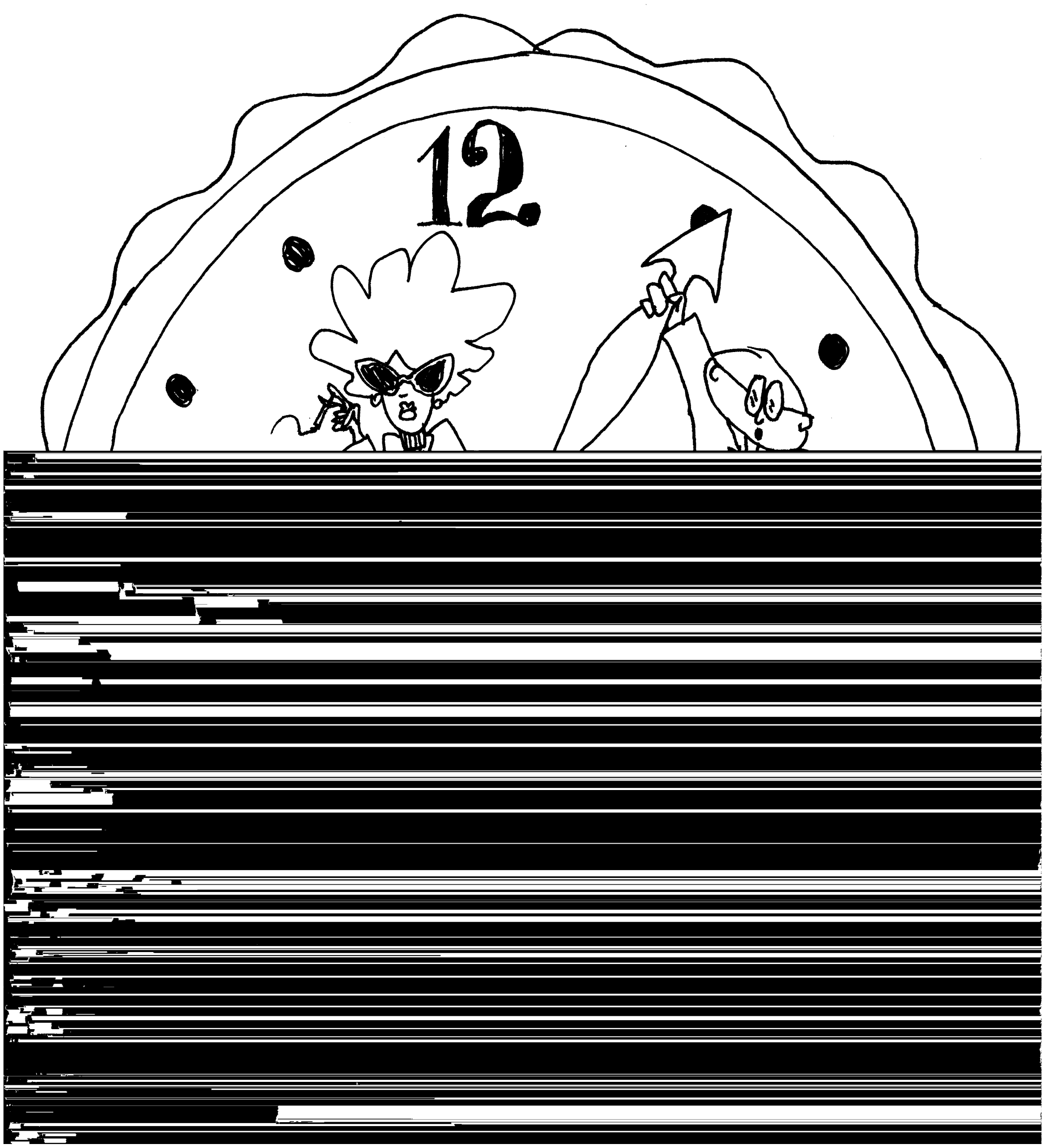
Thanks to all my friends and house mates for putting up ith me.

To my parents, my brother and the grannies, I thank them for their encouragement their support and their interest, for giving me red boxes to play ith, reading to me, and killing the television.

And finally, no thanks to APEC, the Big Cheese, and the Mounties.

Chapter 1

Introduction



1.1 Dual Measurements

One of the first lessons of quantum mechanics as that a property of a system does not correspond to an element of reality until it is measured. It makes no sense to talk about the position of a particle or the momentum of the particle, in and of itself. It is only hen e measure a physical quantity that e can actually say that a system possesses it. The particle does not have a position until its position is actually measured.

Ordinarily in quantum mechanics, one is interested in measuring properties of a system at a particular time t. One might ant to kno a particle's position, momentum, or spin, and the measurement of this quantity occurs at a certain time. For experiments at a fixed time, quantum mechanics provides us ith a useful formalism to describe reality. Observables are represented by self-adjoint operators, and in the Heisenberg representation they evolve in time. The possible results of any measurement at any instant of time t can be found by applying these operators to the ave function of the system at a particular time.

the time becomes the observable one is trying to measure.

Classically, the time of an event can be made into an observable just like any other and this time can be measured in a variety of ays, all of hich give the same result. One can simply invert the equations of motion of the system to find the time that an event occurs [1], and then measure the values of the canonical variables (generalized coordinates and conjugate momenta). Since classically there is no uncertainty relation preventing the measurement of all the coordinates and conjugate momenta simultaneously, there is no limitation for finding the event's time. One could also continually monitor the system to determine the precise time hen the event occurred. Since one can make the interaction bet een the system and the measuring apparatus as small as one likes, this measurement need not disturb the evolution of the system. Finally, one can also couple a clock to the system in such a ay that the clock stops hen the event occurs. All these methods yield the same results, and ork to any desired accuracy.

Dual measurements, are quite common in modern laboratory experiments. In particle physics one often ants to kno the time that certain collisions or decays occurred. Ho ever, surprisingly, dual measurements are not easily dealt ith using the conventional tools of quantum mechanics.

Pauli [8] as the first to demonstrate that there as no operator associated ith time. A time operator must be conjugate to the Hamiltonian, and he proved that this is impossible if the Hamiltonian for the system is bounded from above or belo . The reason for this is that an operator hich is conjugate to the Hamiltonian acts as a shift operator for energy, and one could use it to shift the energy belo any lo er bound (or above any upper bound).

Since then, there have been numerous attempts to circumvent his proof by considering

[1] In some systems (especially in the context of general relativity), it is only possible to find the time locally. A global time variable may not exist.

the time-of-arrival.

The interest in a quantum mechanical time operator stems in part from the troubling

It is space-time hich is the element of reality in general relativity.

These coordinates are, of course, subject to coordinate transformations, and in particular, the theory is invariant under reparametrization of the time coordinate. One consequence of this, is that if one tries to canonically quantize Einstein's theory of gravity in a closed system, one finds that the ave-function must satisfy the Wheeler-DeWitt equation

$$\mathcal{H}\Psi(g_{ab}, \pi_{ab}) = 0 \tag{1.1}$$

here the ave function depends on the 3-metric and conjugate momenta and $\mathcal{H}$ is exist many ambiguities in the role of time in quantum mechanics. Our hope is that a better understanding of time in the arena of quantum mechanics ill benefit and inform research in the field of quantum gravity. At the end of this thesis, e ill discuss some of the connections bet een the problem of time in quantum gravity and our research.

1.2 Differences Between Measurements of Space and Measurements of Time

Ever since Einstein's theory of special relativity, e have been encouraged to think of time and space on an equal footing. Ho ever, even classically, time and space are quite different as our common experience tells us. Objects move constantly for ard in time an a manner very different to the ay they move through space. Although e ill discuss in more detail the differences bet een quantum measurements of ordinary observables and measurements of time in Chapter 2, it may be instructive to roughly outline the differences bet een measurements of a particle's position at a fixed time, and the time a particle is found at a particular location.

In standard quantum mechanics, the probability that a particle is found at a given location X at time t is given by

$$P_t(X) = |\psi(X,t)|^2 \quad . \tag{1.2}$$

If e kno $\psi(x,0)$ for all x then the system is completely described and e can easily compute this probability distribution at an instant of time. If e kno the Hamiltonian of the system, then using the Schrödinger equation e can also compute $\psi(x,t)$ at any time t. This probability distribution corresponds to results of a measurement of position at a particular time. Quantum mechanics gives a ell defined ans er to the question, " here is the particle at time t?"

Ho ever, it is also perfectly natural to ask "at hat time is the particle at a certain location." Here, quantum mechanics does not seem to provide an unambiguous ans er.

At first sight it seems that the probability distribution $P_x(T)$ to find the particle at a certain time at the location x is simply $|\psi(x,T)|^2$, Ho ever, $|\psi(x,T)|^2$, does not represent a probability *in time*, since it is not normalized ith respect to T.

One might be tempted therefore, to consider the quantity

$$P_x(T) = \frac{|\psi(x,T)|^2}{\int |\psi(x,t')|^2 dt'} \tag{1.3}$$

This normalization depends on the particular state being measured, and can only be done if one kno s the state $\psi(x,t)$ at all times t (infinitely far in the past and future). There are also states for hich the particle is never found at the position x, in hich case the expression above is undefined. Not- ithstanding this, one might argue that this quantity gives one a relative probability that the particle is found at the location x at time T (if the measurement is made at that time T), as opposed to another time T' (if the measurement is made at time T').

Ho ever, the expression above certainly does not yield the probability *in time* to detect the particle. One reason for this failure is that a particle may be detected at a location X at many different times t (e.g. I can be found in my office at many different times in the day). On the other hand, if at time t a particle is detected at location X, then e can say ith certainty that at the same time t, the particle as not at any other location X' (e.g. at nine a.m. I am in bed, and therefore, I cannot also be in my office). Equation (1.3) does not give a proper probability distribution as the various outcomes are not disjoint. $P_x(T)$ is not a probability distribution in time in the sense usually reserved for probability distributions in quantum mechanics. $P_x(T)$ is very different from $P_t(X)$ and has different properties (as e ill see in the next chapter).

This leads us to consider the time of first arrival of a particle t log t e rs . M (. T Th the particle as not there at any previous time. In other ords, one must continuously monitor the location x_A in order to find out hen the particle arrives. Ho ever, this continuous measurement procedure has it's o n difficulty, and also emphasizes the problem ith the previous probability distribution. Namely, that the probability to find a particle at $t = T$ is generally *not* independent of the probability to find the particle at some other time $t = T'$. i.e.. if $\mathbf{\Pi}_{x_A}$ is the projector onto the position x_A, then in the Heisenberg representation [3]

$$[\mathbf{\Pi}_{x_A}(t), \mathbf{\Pi}_{x_A}(t')] \neq 0. \tag{1.4}$$

Measurements made at different times disturb each other. We ill see in Section 2.2 that this is one of the properties of ordinary measurements hich measurements in time violate. Measurements made at different times do not commute. Therefore the probability distribution obtained from this measurement procedure, although ell defined, does not give a probability distribution *in time.*

Von Neumann measurements [4] happen *at a certain time.* One measures the particle's position at time t

laar cochati s i a l ts tmes ty

of detecting the particle at some other time t'.

1.3 Inaccuracies and Uncertainties

The measurement of an observable corresponding to a self-adjoint operator can be as accurate as one ishes. This is true despite any uncertainty relations hich govern various sets of observables. The position, or momentum of a particle (but not both) can be measured to any desired precision. Consider t o observables **A** and **B** hich do not evolve in time, and hose commutator is i (in units here $\hbar = 1$). Imagine that e have an ensemble of identical systems prepared in some initial state. On half the ensemble, e can measure **A**, and on the other half, e can measure **B**. Each individual measurement can be as accurate as e ish. An extraordinary experimentalist can reduce the inaccuracies in the measurement to almost zero, and can get a particular value for each measurement. The experimentalist may have a dial on her device hich ill point to the value of A after the measurement. She ill have to make sure that initially the pointer on her dial points almost exactly to zero, and then after each run of her experiment, she measures the position of the dial very accurately to determine the value of A.

If e then plot all of the measurements of **A** and all of the measurements of **B**, e ill find a distribution of measurements hich have a natural idth of ΔA and ΔB respectively. One then finds that no matter hat initial state e choose, $\Delta A \Delta B > 1$. There is an *uncertainty* relation bet een the distributions of A and B, but there are no theoretical limitations on the accuracy of each individual measurement of **A** or **B**.

The experimentalist does not have to make her measurements totally precise. She could, for example, start off the experiment ith her dial in a state here the initial position of the needle is *uncertain*. An uncertainty in the initial pointer position ill result in her measurement being *inaccurate*. When she measures the final position of her

of the probability current to measure the time at hich a particle arrives to a certain location. The discussion suggests that the difference bet een time and other observables is not merely formal.

The central result of the thesis is contained in Chapter 3 here e discuss the problem of the time-of-arrival of a particle to a particular location. It is argued that the time-of-arrival cannot be precisely defined and measured in quantum mechanics. By constructing explicit toy models of a measurement involving physical clocks, e sho that the time-of-arrival for a free particle cannot be measured more accurately then $\delta t_A \sim 1/\bar{E}_k$, here $\bar{E}_k$ is the initial kinetic energy of the particle. With a better accuracy, particles reflect off the measuring device and the resulting probability distribution becomes distorted. This is a ne .

relationship bet een these modified operators, and the direct measurements discussed in Chapters 2 and 3, and argue that a measurement of the time-of-arrival operator does not correspond to these continuous measurements. Unlike the classical case, in quantum mechanics the result of a measurement of the time-of-arrival operator may have nothing to do ith the time-of-arrival to $x = x_A$.

There has been rene ed interest in time-of-arrival operators follo ing the suggestion by Grot, Rovelli, and Tate, that one can modify the lo momentum behavior of the operator slightly in such a ay as to make it self-adjoint [9]. We sho that such a modification results in the difficulty that the eigenstates are drastically altered. In an eigenstate of the modified time-of-arrival operator, the particle, at the predicted time-of-arrival, is found far a ay from the point of arrival ith probability 1/2.

The bound of $1/\bar{E}_k$ on the accuracy of time-of-arrival measurements is based on calculations done using numerous measurement models corresponding to specific Hamiltonians, as ell as more general considerations. Ho ever, because the limitation is based on dynamical considerations and not kinematic ones, a formal proof of the limitation may not exist. For example, a proof of the Heisenberg uncertainty relation relies only on the properties of specific operators, hile our inaccuracy relation is a statement not about operators, but about measurements (and therefore, involves the dynamical considerations of the actual measurement). Perhaps by making certain restrictive assumptions about the Hamiltonian one might be able to construct a formal proof. Such a proof ould have to take into account the measurement model hich ill be discussed in Section 3.3.3 in hich e sho that if one has prior information about the avefunction, and if the avefunction is almost an eigenstate of energy (i.e. its time of arrival is completely uncertain), then one can measure the time of arrival to an accuracy better than $1/\bar{E}_k$. One therefore expects that a formal proof ill not only have to involve making assumptions about the interaction Hamiltonian, but also the initial state of the ave function. The existence of a formal proof for our inaccuracy limitation remains an interesting open question.

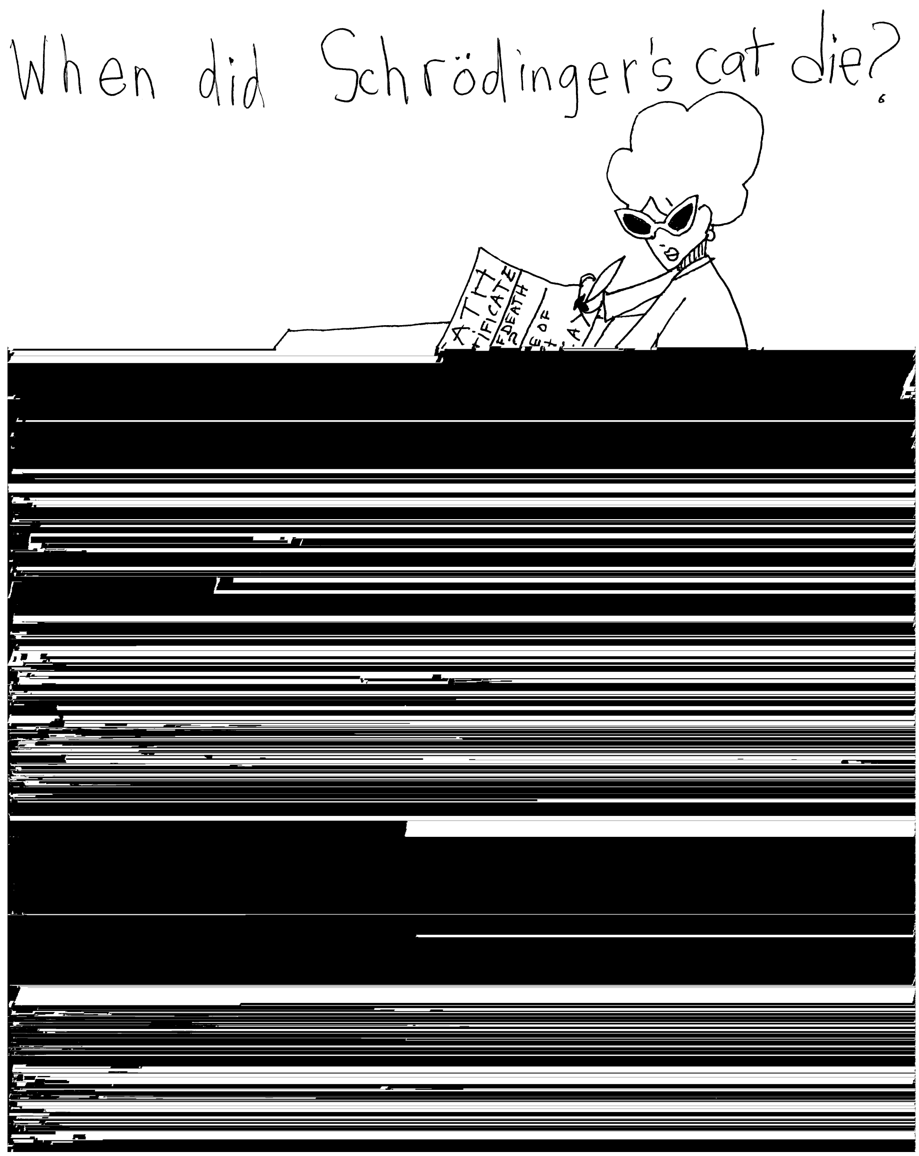
While e kno of no formal proof for the inaccuracy limitation for time-of-arrival, one can make more general statements about measurements of "traversal time". In Chapter 5 e consider the problem of a free particle hich traverses a distance L and argue that a violation of the above limitation for the traversal-time implies a violation of the Heisenberg uncertainty relation for x and p. This result does not depend on the details of the model being used in the measuring process. Measurements of traversal-time are dual to measurements of traversal distance, and it can be sho n that one can measure the distance a particle travels to any desired precision. This chapter also contains a further discussion on the difference bet een hat e call "inaccuracy" limitations, hich constrain the precision ith hich individual measurements are performed, and "uncertainties" hich are kinematic quantities hich relate to the spread in measurements on ensembles.

Chapter 6 contains hat may be our most interesting result. In it, e examine hether one can determine the temporal ordering of events. We find that one cannot measure hether one event occurred in the future or past of another event to arbitrary accuracy. The minimum inaccuracy for measuring hether a particle arrives to a given location before or after another particle is given by $1/\bar{E}$ here $\bar{E}$ is the total kinetic energy of the t o particles. We discuss the relationship bet een this type of measurement, and coincident counters, as ell as Heisenberg's microscope. We sho that in general one cannot prepare a t o particle state here the t o particles al ays arrive ithin a time of $1/\bar{E}$ of each other. This has interesting consequences for determining the metric properties of a space-time.

In this thesis e ill ork in units here $\hbar = c = 1$

Chapter 2

When does an Event Occur



2.1 Probabilities at a Time and in Time

Within quantum mechanics, a complete set of commuting observables can be found hich describe the attributes of a system at a given time. Ho ever, difficulties arise for attributes of a system that extend over time, such as the time of an atomic decay, the time of arrival, etc. As e discuss belo , simple extensions of ordinary notions of probabilities *at a certain time* to probabilities *in time* give rise to distributions hich can no longer be interpreted as probabilities. The reason for this can be understood in simple terms. Consider for example, the event of a particle entering a box. What is the time of the event? Classically, there is no distinction bet een attributes at one time or in time. One can, for example, measure the position and momentum of the particle at any time ith negligible disturbance and use this information to deduce the time of entering the box. Quantum mechanically, ho ever, there are t o separate questions. We can either ask at a certain time t_0, "has the particle already entered the box?", or e can ask " hen did the particle enter the box?"

We ill then examine t o specific cases of measuring the time of an event. One is the arrival of a particle to a certain location, and another is a recent proposal of Rovelli [28] to measure the time that a measurement occurred. We argue that his scheme only ans ers the first question: "has the measurement occurred already at a certain time?", but does not ans er the more difficult question " hen did the measurement occur?" In other ords, it does not provide a proper probability distribution for the time of an event. We also discuss the relationship bet een Rovelli's measurement scheme, and the use of the probability current for measurements of time-of-arrival. In Section 2.4 e discuss a model and set of operators hich can be used to determine a probability distribution for the time of an event. The model is based upon a continuous process akin to a rapid series of measurements. We find that in the limit of high accuracy the system is severely disturbed and the measurement does not ork, an effect hich is analogous to the Zeno paradox.

2.2 Did it Occur vs. When Did it Occur

In conventional quantum mechanics, for each observable, e can assign a set of projection operators $\mathbf{\Pi}_i$ onto a set of eigenstates ϕ_i of some operator. At each time t there exists a Hilbert space and inner product hich enable one to calculate the probability $P_i(t)$ that the system is in one of the states ϕ_i. In certain cases, one can find a subset a of the set i such that the projection operator

F i .. (m. 5o,RT sU given the pbba b llty hat a

$$\mathbf{\Pi}_a = \sum_{i \in}$$

For the case of time-of-arrival, $\mathbf{\Pi}_a$ ill be the projector onto a region of the x-axis (in this case, the index i is continuous).

If initially the system is in the state ψ then in the Heisenberg representation the probability that the event has happened at any time t is given by

$$P_a(t) = \langle\psi|\mathbf{\Pi}_a(t)|\psi\rangle \; . \tag{2.6}$$

One can also compute the "current operator"

$$\mathbf{J}_a = \frac{d\mathbf{\Pi}_a(t)}{dt} \tag{2.7}$$

hich gives the rate of change of the probability distribution $P_a(t)$. It is tempting to argue that the probability distribution

$$p_a(t) = \langle\psi|\mathbf{J}_a(t)|\psi\rangle \tag{2.8}$$

gives the probability that the event a happens bet een t and dt, since classically the probability that an event happened some time before time t is just the integral bet een some initial time t_o and t of the probability that the event happens at that time.

Ho ever, the probability distribution obtained from $\mathbf{J}_a$ cannot be thought of as a probability distribution *in time*. $p_a(t)$ is not the probability that the event happened at time t. To see that $p_a(t)$ is not a probability distribution in time, let us compare its properties to the properties (1-4) of the conventional quantum [1] probability distribution obtained from the projectors $\mathbf{\Pi}_i$.

Property 1 *The probability of finding that the system is in the state ϕ_i at time t is independent of the probability of finding that the system is in the state ϕ_j (at the same time t).*

[1] properties 2-4 are also true of classical probability distributions

i.e..

$$[\mathbf{\Pi}_i(t), \mathbf{\Pi}_j(t)] = 0. \tag{2.9}$$

If e interpret the probabilities $P_a(t)$ as probabilities in time, then our conventional notions of hat these probabilities mean, break do n. In general,

$$[\mathbf{\Pi}_a(t), \mathbf{\Pi}_a(t')] \neq 0. \tag{2.10}$$

Measurements made at earlier times influence measurements made at later times. The possible results of an observable at time t ill depend on hether there ere any previous measurements of that observable. In classical mechanics, one can make the interaction of the measuring device ith the system arbitrarily eak, and therefore, not disturb the evolution of the system in time, but this is not true in quantum mechanics. A measurement of position at t_1 for example, ill disturb the momentum of the particle in such a ay that future measurements of position at t_2 ill give very different results from the case here no measurement as performed at t_1. Since $\mathbf{\Pi}_a(t)$ does not commute ith itself at different times, there is no reason to believe that $\mathbf{J}_a$ ill commute ith itself at different times either. It is essentially this difference bet een conventional probabilities and those obtained from $\mathbf{\Pi}_a$ hich prevents us from determining hen an event occurred.

In addition, $p_a(t)$ and $P_a(t)$ do not have the follo ing other properties of quantum

[ΠΠt) ; Πt)] y θ mK y y m y d θ m y y mfl [RRi m (θ mK different times. There is no reason hy the event a can not happen at many different times. In general

$$\mathbf{\Pi}_a(t)\mathbf{\Pi}_a(t') \neq 0. \tag{2.12}$$

This is also true of classical distributions. For example, I can only be at one place at one time, but I can be at that same place at many different times.

Property 3 *The probabilities $P_i(t)$, are normalized at a given time.*

i.e..

$$\sum_i P_i(t) = 1 \tag{2.13}$$

Ho ever, the operator $\mathbf{J}_a$ is not necessarily normalized in time

$$\int_{-\infty}^{\infty} dt \frac{dP_a(t)}{dt} = \lim_{t->\infty} P_a(t) - P_a(-t). \tag{2.14}$$

In special physical circumstances, $P_a(t)$ may initially be zero, and may finally equal one in the distant future, but there is no reason to expect this to be true in general. One might try to renormalize $\mathbf{J}_a$, but for each initial state the normalization ill in general be different.

This property is also true of classical probability distributions. They also must be normalized, and the classical current is not al ays normalizable. For example, one can have many classical situations in hich the event may never occur. The quantum case is more complicated ho ever, since currents hich are classically positive definate may be negative in the quantum case [29].

Lastly,

Property 4 *The probabilities $P_i(t)$ are positive definite.*

In general, $p_a(t)$ can be negative since $P(t)$ need not be monotonically increasing ith time (this is obviously also true for classical probability functions). One can restrict $\mathbf{J}_a$

to only act on states for hich $\mathbf{P}_a$ is increasing ith time, but the restricted domain of definition of $\mathbf{J}_a$ may mean that it ill no longer be self-adjoint. Furthermore, hether $\mathbf{J}_a$ is positive or negative ill not only depend on the state, but also on the Hamiltonian. For certain Hamiltonians, one may find that there are no states for hich $p_a(t)$ does not take on negative values.

Another interesting aspect of $\mathbf{J}_a$ and $\mathbf{\Pi}_a$ is that in general

$$[\mathbf{J}_a(t), \mathbf{\Pi}_a(t)] \neq 0. \tag{2.15}$$

The operator hich measures that the event happened and the operator $\mathbf{J}_a$ do not commute. If one believes that $\mathbf{J}_a$ can be used to ans er the question " hen did the event happen?" then one finds that " hen did it happen?" and "has it already happened?" seem to be complimentary (in Bohr's sense) in that they interfere ith each other. Naively, it ould seem that determining " hen did a occur?" ould also ans er the question "has a occurred?". Ho ever the inaccuracy of the determination of " hen did a occur?" seems to place limits on our ability to ans er "has a occurred?".

2.3 Time of a Measurement or Arrival

We no examine t o specific examples of the determination of hen an event occurred. In the first example, one tries to determine hen a measurement occurred. In the second example, one ishes to determine the time at hich a particle arrives to $x = 0$

Let us try to measure the time that a measurement occurred (a measurement of a measurement in a sense). Imagine that e ant to find out the time of a measurement of the observable $\mathbf{A}$ of a quantum system S. The measurement of $\mathbf{A}$ can be accomplished by coupling a macroscopic apparatus O to the system, via an ry grmt ok here $\mathbf{P}$ is the conjugate momentum to the pointer $\mathbf{Q}$ of the measuring device, and $g(t)$ is a function hich is zero every here, except during a small interval of time. After the measurement is complete, the measuring apparatus ill be correlated ith the state of the system. If initially, S is in a superposition of eigenstates $|\phi_i\rangle$ of the observable $\mathbf{A}$, so that $|\psi_S\rangle = \sum_i c_i|\phi_i\rangle$, then e expect the initial state of the combined $S - O$ system to evolve into a correlated state.

$$\sum_i c_i|\phi_i\rangle \otimes |O\rangle \to \sum_i c_i|\phi_i\rangle \otimes |O_i\rangle \tag{2.17}$$

here $|O\rangle$ is the original state of the device and the $|O_i\rangle$ are orthogonal states of the measuring apparatus hich are correlated ith the system. If the coupling is small, then the duration of the measurement might need to be long in order to distinguish bet een the various eigenvalues of $\mathbf{A}$. At any time during the measurement, it is possible to calculate the density matrix of the combined S-O system. One can imagine that a second apparatus O' measures the state of the first apparatus O to determine hether a measurement has occurred. This has been studied for the case hen the measurement is gradual [21] [27]. Rovelli [28] has recently proposed that the apparatus O' might measure the operator

$$\mathbf{M} = \sum_i |\phi_i\rangle \otimes |O_i\rangle\langle O_i| \otimes \langle\phi_i|$$

In the case of time-of-arrival, one ishes to measure the time a particle arrives to a certain location (say $x = 0$). Often, the probability current is used to determine the arrival time[13]. One imagines that a particle is localized in the region $x < 0$ and traveling to ards the origin. The projector

$$\mathbf{\Pi}_+ = \int_0^\infty dx|x\rangle\langle x| \tag{2.21}$$

is an operator hich is equal to one hen $x > 0$ and zero other ise. The probability of detecting the particle in the positive x-axis is given by

$$P_+(t) = \langle\psi|\mathbf{\Pi}_+(t)|\psi\rangle. \tag{2.22}$$

In the Schrödinger representation, this expression is just $P_+(t) = \int_0^\infty |\psi(x,t)|^2 dx$. It is then claimed that the current $\mathbf{J}_+$, given by

$$\frac{\partial \mathbf{J}_+}{\partial x} = \frac{d\mathbf{\Pi}_+(t)}{dt} \tag{2.23}$$

ill give the probability that the particle arrives bet een t and $t + dt$.

It is clear that both the operators $\mathbf{M}(t)$ and $\mathbf{\Pi}_+(t)$ are specific example of the operator $\mathbf{\Pi}_a$ discussed in Section 2.2. $\mathbf{M}(t)$ gives the probability at time t that a measurement has occurred. $\mathbf{\Pi}_+(t)$ gives the probability that the particle is found at $x > 0$ at time t. The t o operators $\mathbf{m}(t)$ and $\frac{\partial \mathbf{J}_+(t)}{\partial x}$ are examples of $\mathbf{J}_a(t)$. $\mathbf{m}(t)$ gives the change in the probability that the measurement happened at time t, hile $\frac{\partial \mathbf{J}_+(t)}{\partial x}$ gives the change in the probability that the particle is found at $x > 0$. Ho ever, one cannot interpret these operators as giving the probability that the measurement occurred (or the particle arrived). None of these operators allo one to measure the precise time at hich the event occurred. They do not posses all the Properties 1-4 listed above.

Considered as probabilities in time, none of the operators above give distributions hich have Property 1. The operators above do not commute ith the Hamiltonian, and therefore depend on t. For $t - t' \ll d\mathbf{H}$ e have for any operator $\mathbf{A}(t) \simeq \mathbf{A}(t') + i(t - t')[\mathbf{H}, \mathbf{A}(t')]$, and so

$$[\mathbf{A}(t), \mathbf{A}(t')] \simeq i(t - t')[\mathbf{H}, \mathbf{A}(t')], \mathbf{A}(t')] \tag{2.24}$$

For arbitrary Hamiltonians, it is obvious that none of the operators above ill commute ith themselves at different times. Even for a free particle, one can explicitly calculate that neither $\frac{\partial \mathbf{J}_+(t)}{\partial x}$ nor $\mathbf{\Pi}_+(t)$ commute ith themselves at different times, (the calculation is neither difficult, nor particularly illuminating).

For some very specific states, and physical situations, Properties 2-4 may be obeyed, but this is certainly not true in general. For the case of time-of-arrival, even for a free Hamiltonian and ave packets hich only contain modes of positive frequency, the current can be negative [29] (a violation of Property 4 - that probabilities must be positive definite). In fact, since the current is simply the time derivative of a projection operator, there is no reason to expect it to al ays be positive. For free particles hich can arrive from the left and right, the current can be zero [2] and hence the probability distribution ill be unnormalizable (Property 3). Also, in general, there is no reason hy a particle can't be at the same position at many different times (a violation of Property 2). In the case of particles hich move in a potential, one may find that there are no states for hich Properties 3-4 are obeyed. For example, if there is an infinite potential barrier around the origin, the particle ill never arrive, and the current ill not be normalizable, and if there is a harmonic oscillator potential, the particle ill cross the origin many times from both the left and the right violating Properties 2 and 4. For the case of determining hen a measurement occurred, Rovelli restricts the class of measurements he considers to be those for hich $\mathbf{m}(t)$ obeys Properties 2-4. As a result, $\mathbf{m}(t)$ cannot be used for arbitrary measurements. As ith the time-of-arrival, there are clearly many Hamiltonians

[2]See Appendix A where we see that a coherent antisymmetric superposition of left and right moving waves has zero current.

for hich Property 2-4 ill be violated. Nor can $\mathbf{m}(t)$ be used for Hamiltonians for hich its restricted domain of definition ill mean that it is no longer self-adjoint.

Although operators such as $\mathbf{m}$ and and $\mathbf{J}_+$ do not commute ith themselves at different times, it is possible to construct an operator hich is time-translation invariant, and ould give the time of an event in the classical limit. This ill be discussed in Chapter 4 here e shall see that such an operator cannot be self-adjoint if the Hamiltonian is bounded from above or belo .

2.4 Continual Event Monitoring

Instead of considering operators, a more physical meaningful method of measuring the occurrence of an event is to consider continuous measurement processes. For example, the operator $\mathbf{\Pi}_a(t)$ can be measured continuously or at small time intervals. When one considers such a physical measurement procedure one can see that the time at hich an event occurs is not ell defined in quantum mechanics. The probability of finding that the system enters one of the states ϕ_i at time t_a is given by the probability that it isn't in any of the states ϕ_i before t_a, times the probability that it is in one of the states ϕ_i at t_a.

To see ho such a scheme might ork, let us see ho one ould measure the time of an occurrence of the event corresponding to $\mathbf{\Pi}_a$. A measurement of the operator $\mathbf{\Pi}_a(t)$ ill tell us hether the event has occurred at time t. We can then measure $\mathbf{\Pi}_a(t)$ at times $t_k = k\Delta$ for integral k in order to determine hen the measurement occurred. Δ represents the frequency ith hich e monitor the system, and is therefore the inaccuracy of the measurement in time (it is the coarseness of the measurement in some sense).

We ill no ork in the Schrödinger representation, simply because it is the most natural arena to talk about successive measurements on a system. At time t_1, the probability that an event has occurred is given by

$$P(\uparrow, t_1) = \langle\psi_0(0)|\mathbf{U_\Delta}^\dagger \mathbf{\Pi}_a \mathbf{U_\Delta}|\psi_0(0)\rangle \tag{2.25}$$

and the probability that it hasn't is

$$P(\downarrow, t_1) = \langle\psi_0(0)|\mathbf{U_\Delta}^\dagger (\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0(0)\rangle \tag{2.26}$$

here $\uparrow$ corresponds to detecting that the event has occurred, $\downarrow$ corresponds to detecting that an event has not yet occurred, $\psi_0(0)$ is the initial state of the system and $\mathbf{U_\Delta}$ is the evolution operator $e^{-i\mathbf{H}\Delta}$. If the result is $\downarrow$, e collapse the ave function and evolve it to the next instant. The normalized state before the second measurement is:

$$|\psi_2(t_2)\rangle = \frac{\mathbf{U_\Delta}(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0\rangle}{\langle\psi_0|\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0\rangle^{1/2}} \tag{2.27}$$

The probability that an event has occurred at t_2 is given by the probability that an event didn't occur at t_1 times the probability that ψ_2 is in one of the states ϕ_i

$$P(\uparrow, t_2) = \frac{\langle\psi_0|\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}^\dagger \mathbf{\Pi}_a \mathbf{U_\Delta}(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0\rangle}{\langle\psi_0|\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0\rangle} \times \langle\psi_0|\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0\rangle \tag{2.28}$$

The probability that an event didn't occur is given by

$$P(\downarrow, t_2) = \langle\psi_0|\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}|\psi_0\rangle \tag{2.29}$$

By repeating this process, e find that at time t_k the probability that an event has occurred is given by

$$P(\uparrow, t_k) = \langle\psi_0|A_k|\psi_0\rangle \tag{2.30}$$

here

$$A_k = \mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)...\mathbf{U_\Delta}^\dagger \mathbf{\Pi}_a \mathbf{U_\Delta}...(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta} \tag{2.31}$$

and the probability that an event hasn't occurred is

$$P(\downarrow, t_k) = \langle\psi_0|B_k|\psi_0\rangle \tag{2.32}$$

ith

$$B_k = \mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)...\mathbf{U_\Delta}^\dagger(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}...(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta}(\mathbf{1} - \mathbf{\Pi}_a)\mathbf{U_\Delta} \tag{2.33}$$

By allo ing the unitary operators to act on the projection operators e can rite the A_k or B_k in the Heisenberg representation. For example

$$A_k = (\mathbf{1} - \mathbf{\Pi}_a)(t_1)...(\mathbf{1} - \mathbf{\Pi}_a)(t_{k-1})\mathbf{\Pi}_a(t_k)(\mathbf{1} - \mathbf{\Pi}_a)(t_{k-1})...(\mathbf{1} - \mathbf{\Pi}_a)(t_1) \tag{2.34}$$

Ho ever, hile the operators $\mathbf{\Pi}_a(t)$ can be found by unitary time-evolution of $\mathbf{\Pi}_a(0)$, the operators A_k and B_k are not related by a unitary transformation to A_0 and B_0. This already signals that they can not give an undisturbed distribution for the time of an event. Nor are the A_k and B_k projection operators.

The probabilities derived from A_k and B_k are not universal. In this case, they apply only to the specific measurement scenario under discussion. In particular the probability distribution is sensitive to the frequency at hich $\mathbf{\Pi}_a$ is measured, a phenomenon hich is related to the Zeno paradox [23].

As an example, consider a measurement of the spin of a particle. We ish to kno at hat time the measurement occurred. The particle is in a state given by

$$|\psi_S\rangle = a|\uparrow\rangle + b|\downarrow\rangle \tag{2.35}$$

and e use a simple measuring device hich is also a spin 1/2 particle initially in the state $|O\rangle = |\uparrow'\rangle$, hich evolves according to the Hamiltonian

$$\mathbf{H} = g(t)\sigma'_x\frac{1}{2}(1 - \sigma_z) \tag{2.36}$$

here $\int g(t)dt = \pi$ ($g(t)$ is sharply peaked, ith idth T), and the primed Pauli matrix acts on the measuring device, hile the unprimed Pauli matrix acts on the system. After a time T, the spin of the measuring device ill be correlated ith the system. Since this measurement is rather crude, (the initial state of the device is the same as one of the measurement states), the operator $\mathbf{M}$ at $t = 0$ is not zero. Let us simplify the problem further, by assuming that $a = 0$ and $b = 1$. In this case, the only relevant matrix element of $\mathbf{1} - \mathbf{M}$ is $|\downarrow\rangle \otimes |\uparrow'\rangle\langle\uparrow'| \otimes \langle\downarrow| = |\psi_o\rangle\langle\psi_o|$. We then find the probability that the measuring apparatus has not responded at time t_k is

$$
\begin{aligned}
P(\downarrow, t_k) &= |\langle\psi_o|U_\Delta|\psi_o\rangle|^{2k} \\
&\simeq |\langle\psi_o|1 - i\Delta\mathbf{H} - \Delta^2\mathbf{H}^2|\psi_o\rangle|^{2k} \\
&\simeq 1 - \Delta^{2k}(\langle\mathbf{H}^2\rangle - \langle\mathbf{H}\rangle^2)^k
\end{aligned}
\tag{2.37}
$$

If e fix a value of $\tau = t_k$ and then make Δ go to zero, e find

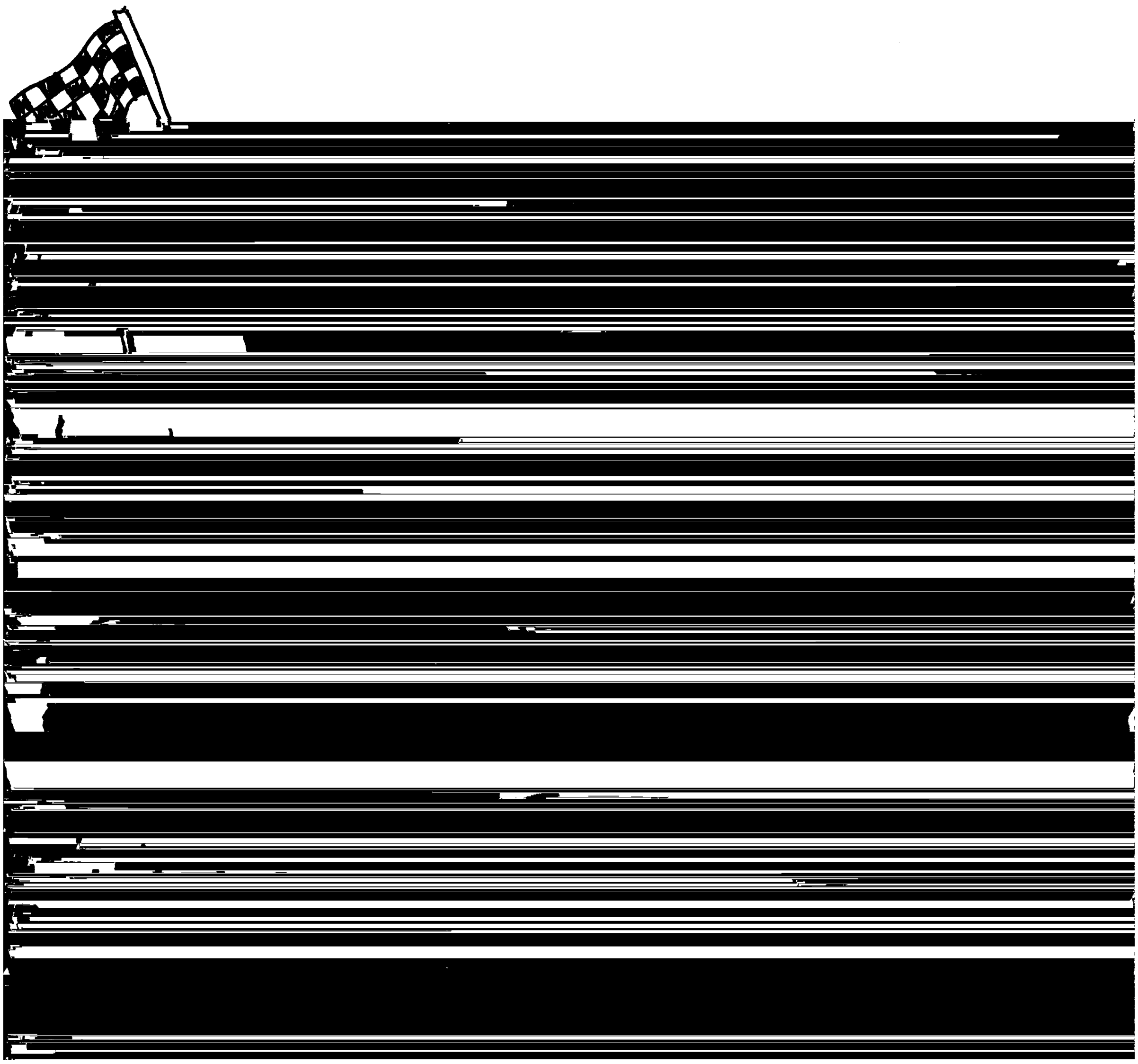
$$
\begin{aligned}
P(\downarrow, \tau) &\simeq e^{-(\Delta dE)^{2\tau/\Delta}} \\
&\simeq 1,
\end{aligned}
\tag{2.38}
$$

hich implies that the measuring apparatus becomes frozen and never records a measurement. In order not to freeze the apparatus, e need $\Delta > 1/dE$ here dE is the uncertainty in energy of the measuring device O (initially the spacing bet een energy levels in this case). There is al ays an inherent inaccuracy hen measuring the time that the event (of the measurement) occurred. This inaccuracy is similar to the one hich e ill find in Chapter 3. Note that as discussed in the Introduction, this inaccuracy is not related to the so-called

One can of course use the set of operators A_k to compute a probability distribution in time, or experimentally determine a probability distribution for the time of an event. Ho ever, as e have just seen, this probability distribution is not a function of the system alone, but rather, it is related to the system and the measuring device (or set of operators) For example, the probability distribution ill depend on Δ, and if Δ is too small, e ill find that the event never occurs. The distribution $P(\uparrow, t_k)$ does allo as to predict the probabilities of future meɑct the pc hes lly ds u ties m ;M ;u ter ice s bit o f

Chapter 3

Physical Clocks and Time-of-Arrival



3.1 A Limitation on Time-of-Arrival Measurements

In the previous Chapter, e sa that if e attempt to measure the time of an event using a rapid series of measurements, then our measurement ill disturb the very thing hich e are trying to measure. For the simple case of a t o state system, e ere able to sho that if e make the measurement accurately enough, the system freezes, and the event never occurs.

In this Chapter, e study measurements of the time-of-arrival of a particle to a particular location using physical clocks. The clocks are coupled to the system in such a ay that hen the particle arrives to the fixed location, the clock ill read the time-of-arrival. Unlike the continuous measurement procedure discussed in the last Chapter, e obtain the time of the event using only a single measurement hich is made ell after the event has occurred. Nonetheless, e ill find that if e make the measurement extremely accurate, the measurement ill fail.

Consider a free particle, upon hich a measurement is performed to determine the time-of-arrival to $x = x_A$. The time-of-arrival can be recorded by a clock situated at $x = x_A$ hich s itches off hen the particle reaches it. In classical mechanics e could, in principle, achieve this ith the smallest non-vanishing interaction bet een the particle and the clock, and hence measure the time-of-arrival ith arbitrary accuracy.

In classical mechanics there are other indirect methods to measure the time-of-arrival. One could invert the equation of motion of the particle and obtain the time in terms of the location and momentum, $T_A(x(t), p(t), x_A)$. This function can be determined at *any time* t, either by a simultaneous measurement of $x(t)$ and $p(t)$ and evaluation of T_A, or by a direct coupling to $T_A(x(t), p(t), x_A)$. One could also measure the time-of-arrival using the method discussed in the previous chapter. By using a eak interaction hich doesn't disturb the system, one can continually monitor the point of arrival to see if the particle

From (3.41) e see that in order to use this clock to read the time, e need to kno the initial position of the clock's dial $\mathbf{y}(t_0)$ and then subtract this from our final reading of $\mathbf{y}$. Quantum mechanics puts no limitation on ho accurately this clock can be measured. If e ant to accurately infer the time from the final reading of the clock then the clock must initially be prepared in a state ith a very small uncertainty in y. At some later point, e can measures the coordinate $y(t_f)$ to any degree of accuracy e ish to infer the time from $y($

of energy to turn them off. To measure the time-of-arrival of a particle, the particle itself ill have to turn off the clock hen it arrives – the external observer cannot supply any energy since she does not kno hen to turn the clock off. If the clock is much more energetic than the particle, then it ill be impossible for the particle to turn off the clock and no record ill exist that the particle arrived. In fact,

Hence at $x = x_{peak}$ the clock coordinate y is peaked at the classical time-of-arrival

$$y = \frac{mx_o}{k_0}. \tag{3.57}$$

To see that the clock yields a reasonable record of the time-of-arrival, let us consider further the probability distribution of the clock

$$\rho(y, y)_{x>0} = \int dx |\psi(x > 0, y, t)|^2. \tag{3.58}$$

In the case of inaccurate measurements ith a small back-reaction on the particle $A_{I'} \simeq 1$. The clocks density matrix is then found (see Appendix B) to be given by:

$$\rho(y, y)_{>0} \simeq \frac{1}{\sqrt{2\pi\gamma(y)}} e^{-(y}$$

Since the possible values obtained by p are of the order $1/\Delta_y \equiv 1/\Delta t_A$, the probability to trigger the clock remains of order one only if

$$\bar{E}_k \delta t_A > 1. \tag{3.61}$$

Here δt_A stands for the initial uncertainty in position of the dial $\mathbf{y}$ of the clock, and is interpreted as the accuracy of the clock. $\bar{E}_k$ can be taken as the typical initial kinetic energy of the particle.

In measurements ith accuracy better then $1/\bar{E}_k$ the probability to succeed drops to zero like $\sqrt{E_k \delta t_A}$, and the time-of-arrival of most of the particles cannot be detected. Furthermore, the probability distribution of the fraction hich has been detected depends on the accuracy δt_A and can become distorted ith increased accuracy. This observation becomes apparent in the follo ing simple example. Consider an initial ave packet that is composed of a superposition of t o Gaussians centered around $k = k_1$ and $k = k_2 >> k_1$.

trigger ithout including the clock:

$$H_{trigger} = \frac{1}{2m}\mathbf{P}_{\mathbf{x}}^{\mathbf{2}} + \frac{\alpha}{2}(1+\sigma_x)\delta(\mathbf{x}). \tag{3.62}$$

The particle interacts ith the repulsive Dirac delta function potential at $x = 0$, only if the spin is in the $|\uparrow_x\rangle$ state, or ith a vanishing potential if the state is $|\downarrow_x\rangle$. In the limit $\alpha \to \infty$ the potential becomes totally reflective (Alternatively, one could have considered a barrier of height α^2 and idth $1/\alpha$.) In this limit, consider a state of an incoming particle and the trigger in the "on" state: $|\psi\rangle|\uparrow_z\rangle$. This state evolves to

$$|\psi\rangle|\uparrow_z\rangle \quad \to \quad \frac{1}{\sqrt{2}}\Big[|\psi_R\rangle|\uparrow_x\rangle + |\psi_{T}\rangle|\downarrow_x\rangle\Big], \tag{3.63}$$

here ψ_R and ψ_T are the reflected and transmitted ave functions of the particle, respectively.

The latter equation can be re ritten as

$$\frac{1}{2}|\uparrow_z\rangle(|\psi_R\rangle + |\psi_T\rangle) + \frac{1}{2}|\downarrow_z\rangle(|\psi_R\rangle - |\psi_T\rangle) \tag{3.64}$$

Since $\uparrow_z$ denotes the "on" state of the trigger, and $\downarrow_z$ denotes the "off" state, e have flipped the trigger from the "on" state to the "off" state ith probability 1/2 [2]. Although this model only orks half the time, the chance of success does not depend in any ay on the system, and in particular, on the particle's energy. Furthermore, one can construct models here a detector is triggered almost all the time [35], although ith some energy dependence in the probability of triggering.

So far e have succeeded in recording the event of arrival to a point. We have no information at all on the time-of-arrival. It is also orth noting that the net energy exchange bet een the trigger and the particle is zero, i.e.. the particle's energy is unchanged.

[2]It is interesting to see that for some wave functions which represent coherent superpositions of particles arriving from both the left and right, the detector is never triggered. An example of this is given in Appendix A.

transmitted aves.

The eigenstates of (3.73), in the basis of σ_z, are given by

$$\Psi$$

There is ho ever one limiting case in hich the method does seem to succeed. Consider a narro ave peaked around k ith a idth dk. To first order in dk, the probability $T_\downarrow$ that the particle is successfully boosted is given by

$$T_\downarrow \simeq 1 - \frac{2dk}{k}. \tag{3.80}$$

Therefore in the special case that $\frac{dk}{k} << 1$, the transition probability is still close to one. If in this case e kno n in advance the value of k up to $dk << k$, e can indeed use the booster to improve the bound (3.61) on the accuracy.

The reason hy this seems to ork in this limiting case is as follo s. The probability of flipping the particle's spin depends on ho long it spends in the magnetic field described by the α term in (3.73). If ho ever, e kno beforehand, ho long the particle ill be in this field, then e can tune the strength of the magnetic field (α) so that the spin gets flipped. The requirement that $dk/k << 1$ is thus equivalent to having a small uncertainty in the "interaction time" ith this field. In some sense, the measurement is possible, because e kno the particle's momentum before hand. Of course, if e have prior kno ledge of the particle's momentum, then e could just measure $\mathbf{x}$ and argue that this allo s us to calculate the time of arrival through $t_A = mx/p$. We therefore believe that the reason the measurement procedure described above orks in this limiting case is because it assumes prior kno ledge of the particle's momentum, and e do not believe that one could improve it so that it orks for all states. These "booster" measurements cannot be used for general ave functions, and even in the special case above, one still requires some prior information of the incoming ave function.

3.3.4 Gradual triggering of the clock

In order to avoid the reflection found in the previous t o models, e shall no replace the sharp step-function interaction bet een the clock and particle by a more gradual transition.

When the WKB condition is satisfied

$$\frac{d\lambda(x)}{dx} = \epsilon << 1 \tag{3.81}$$

here $\lambda(x)^{-2} = 2m[E_0 - V(x)]$, the reflection amplitude vanishes as

$$\sim \exp(-1/\epsilon^2) \tag{3.82}$$

Solving the equation for the potential ith a given ϵ e obtain

$$V_\epsilon(x) = E_0 - \frac{1}{2m\epsilon^2}\frac{1}{x^2} \tag{3.83}$$

No e observe that any particle ith $E \geq E_0$ also satisfies the WKB condition (3.81) above for the *same* potential V_ϵ. Furthermore $p_y V_\epsilon$ also satisfies the condition for any $p_y > 1$.

These considerations suggest that e should replace the Hamiltonian in eq. (3.49) ith

$$H = \mathbf{P_x}^2/2m + V(x)\mathbf{P_y} \tag{3.84}$$

here

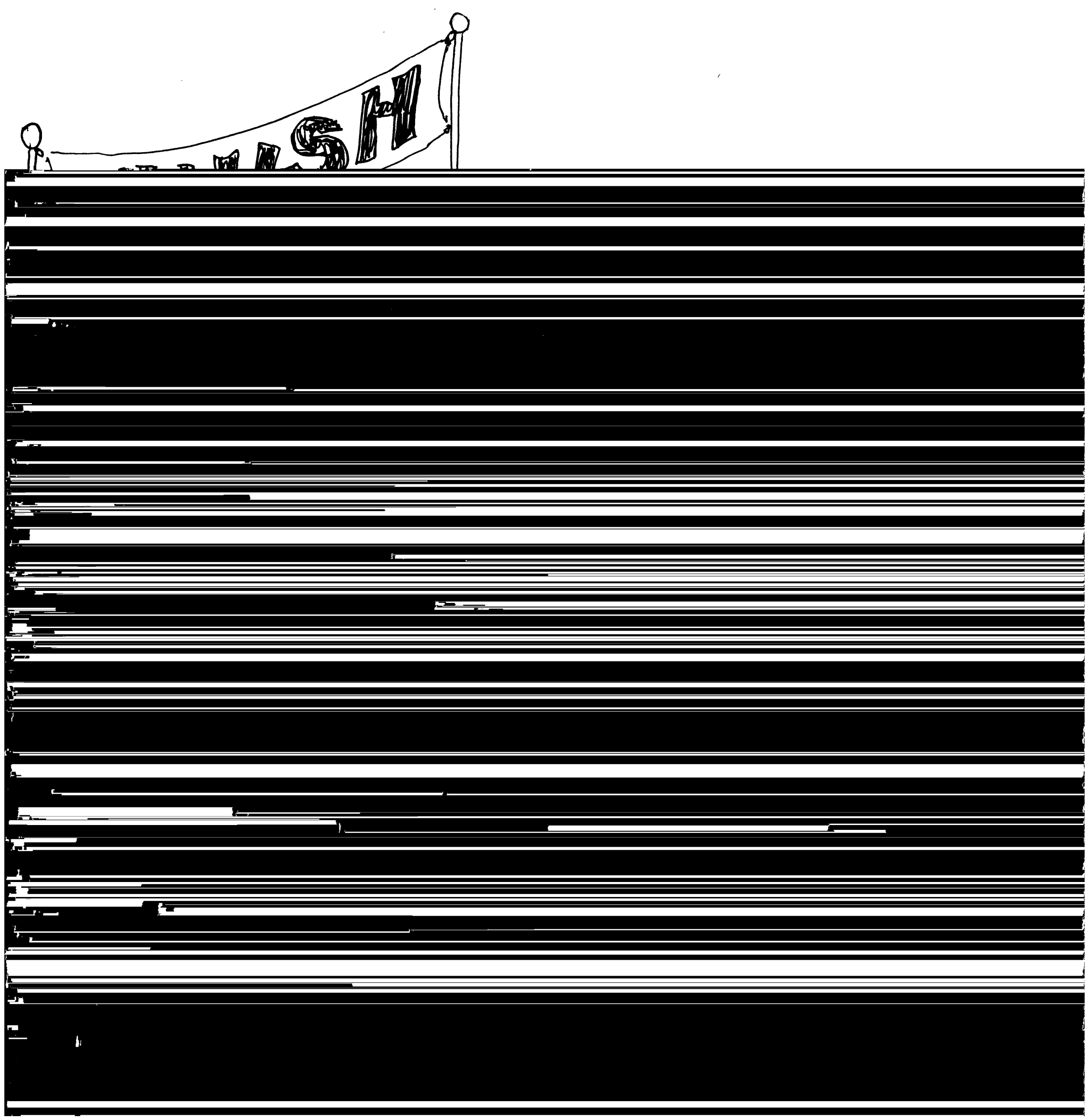
$$V(x) = \begin{cases} -\frac{x_A^2}{x^2} & x < x_A \\ -1 & x \geq x_A \end{cases} \tag{3.85}$$

Here $x_A^{-2} = 2m\epsilon^2$.

Thus this model describes a gradual triggering *on* of the clock hich takes place hen the particles propagates from $x \to -\infty$ to ards $x = x_A$. In this case the arrival time is approximately given by $t - \mathbf{y}$ here $t = t_f - t_i$. Since ithout limiting the accuracy of the clock e can demand that $p_y >> 1$, the reflection amplitude off the potential step is exponentially small for *any* initial kinetic energy E_k.

Chapter 4

Time-of-Arrival Operators



4.1 Indirect Time-of-Arrival Measurements

In the previous chapter, e sa that one cannot measure the time-of-arrival of a free particle to arbitrary accuracy by coupling the particle to a clock. Still, one can imagine an indirect determination of arrival time by a measurement of some regularized time-of-arrival operator $\mathbf{T}(\mathbf{x}(t), \mathbf{p}(t), x_A)$ [9]. In quantum mechanics, ordinary observables like position and momentum *are* represented by operators at a fixed time t. Ho ever, e ill sho that there is no operator associated ith the time it takes for a particle to arrive to a fixed location. In Section 4.2 e ill prove formally that in general a Hermitian time-of-arrival operator ith a continuous spectrum can only exist for systems ith an unbounded Hamiltonian. This is because the existence of a time-of-arrival operator the point of arrival ith probability 1/2. We also calculate the average energy of the states, in order to relate them to our proposal in Chapter 3 that one cannot measure the time-of-arrival to an accuracy better than $1/\bar{E}_k$. We end ith concluding remarks in Section 4.7.

4.2 Conditions on A Time-of-Arrival Operator

As discussed in the previous section, although a direct measurement of the time-of-arrival may not be possible, one can still try to observe it indirectly by measuring some operator $\mathbf{T}(\mathbf{p}, \mathbf{x}, x_A)$. In the next t o sections e shall examine this operator and its relation to

that equation (4.103) is inconsistent unless the Hamiltonian is unbounded from above and belo [8].

4.3 Time-of-Arrival Operators vs. Continuous Measurements

Although formally there cannot exist a time-of-arrival operator $\mathbf{T}$, it may be possible to approximate $\mathbf{T}$ to arbitrary accuracy [9]. This modified operator ill be discussed more fully, in the next section, but for no , assume that e can define the regularized Hermitian operator

$$\mathbf{T}' = O(\mathbf{p}) T \mathbf{O}(\mathbf{p}) \tag{4.104}$$

here $O(\mathbf{p})$ is a function hich is equal to 1 at all values of p except around a small neighborhood of $k = 0$. For $|p| < \epsilon$, $O(\mathbf{p})$ goes rapidly to zero (at least as fast as $\sqrt{k}$). $\mathbf{T}'$ is thus an operator hich behaves just like $\mathbf{T}$ except in a very small neighborhood conjugate to H. The value of $\mathbf{T}'$ is recorded on the conjugate of q – call it P_q. No the uncertainty is given by $dT'_A = d(P_q) = 1/dq$, thus naively from $dq = 1/dT'_A < E_{min}$, e get $E_{min}dT' > 1$. Ho ever here, the average $\langle q \rangle$ as taken to be zero. There is no reason not to take $\langle q \rangle$ to be much larger than E_{min}, so that $\langle q \rangle - dq >> -E_{min}$. If e do so, the measurement increases the energy of ψ and $\mathbf{T}'$ is al ays conjugate to H. The limitation on the accuracy is in this case $dT'_A > 1/\langle q \rangle$ hich can be made as small as e like.

Ho ever, even small deviations from the commutation relation (4.103) are problematic. Not only is the modificati . [illegible] larger than E [illegible] E a h one needs to kno the full Hamiltonian from time t' until t_A. Even if one kno s the full Hamiltonian, and can find an approximate time-of-arrival operator, one has to have faith that the Hamiltonian ill not be perturbed after the measurement has been made. On the other hand, the continuous measurements e have described can be used ith any Hamiltonian.

A further difficulty is that a measurement of the time-of-arrival operator is not equivalent to continuously monitoring the point-of-arrival I.e., measuring the time-arrival-operator is not equivalent to the measurement procedures discussed in Chapter 2. If $\mathbf{P}_0$ is the projector onto $x = 0$, one finds that

$$\langle\psi[\mathbf{T}, \mathbf{P}_0]|\psi\rangle = -\frac{i}{2}Re\{\psi(x=0)\int dk\psi^*(k)\frac{m}{k^2}\}. \qquad (4.110)$$

A measurement of the time-of-arrival operator does not commute ith the projection operator onto the point of arrival.

It is also clear from the discussion in Section 3.3, that in the limit of high precision, continuous measurements respond very differently to the time operator. When measurements are made ith physical clocks, then in the limit $dt_A \to 0$ all the particles bounce state of the particle if the clock has stopped, and $|\chi'(t_A)\rangle$ the final state of the particle if the clock has not stopped.

Since the eigenstates of **T** form a complete set, e can express any state of the particle as $|\psi\rangle = \int dt_A C(t_A)|t_A\rangle$. We then obtain :

$$\int dt_A C(t_A)|t_A\rangle|y=t_0\rangle \ \rightarrow \ \int dt_A C(t_A)|\chi(t_A)\rangle|y=t_A\rangle + \left(\int dt_A C(t_A)|\chi'(t_A)\rangle\right)|y=t\rangle. \tag{4.112}$$

The final probability to measure the time-of-arrival is hence $\int dt_a|C(t_a)\chi(t_a)|^2$. On the other hand e found that for a general ave function ψ, in the limit of $dt_a \rightarrow 0$, the probability for detection vanishes. Since the states of the clock, $|y=t_a\rangle$, are orthogonal in this limit, this implies that $\chi(t_a)=0$ in eq. (4.111) for all t_A. Therefore, the eigenstates of **T** cannot trigger the clock.

It should be mentioned ho ever, that one ay of circumventing this difficulty may be to consider a coherent set of **T** eigenstates states instead of the eigenstates themselves. These normalizable states ill no longer be orthogonal to each, so they may trigger the clock if they have sufficient energy [1]. In this regard it is of interest to prematurely quote a result hich e ill sho in Section 4.5 - the average energy of a Gaussian distribution of time-of- arrival eigenstates is proportional to $1/\Delta$ here Δ is the spread of the Gaussian. This puts us at the edge of the limitation given in Equation (3.91).

[1] An arbitrary wave packet can be written as a superposition of normalized eigenstates, and yet we know that arbitrary wave packets do not trigger the clocks of Section 3.3.1. This creates a somewhat interesting situation if normalized eigenstates trigger a clock, but wave packets made of superpositions of them do not.

4.4 The Modified Time-of-Arrival Operator

Kinematically, one expects that the time-of-arrival operator for a free particle arriving at the location $x_A = 0$ might be given by

$$\mathbf{T} = -\frac{m}{2}\frac{1}{\sqrt{\mathbf{p}}}\mathbf{x}(0)\frac{1}{\sqrt{\mathbf{p}}}. \tag{4.113}$$

The operator $-m(\frac{1}{\mathbf{p}}\mathbf{x} + \mathbf{x}\frac{1}{\mathbf{p}})$ is equivelant to the one above as can be seen by use of the commutation relations for $\mathbf{x}$ and $\mathbf{p}$.

In the k representation this operator can be ritten as

$$\mathbf{T}(k) = -im\frac{1}{\sqrt{k}}\frac{d}{dk}\frac{1}{\sqrt{k}} \quad = -im(\frac{1}{k}\frac{d}{dk} + \frac{d}{dk}\frac{1}{k}) \tag{4.114}$$

 here $\sqrt{k} = i\sqrt{|k|}$ for $k < 0$. If one solves the eigenvalue equation, one finds a set of anti-symmetric states [2] for this operator given by

$$g^a_{t_A}(k) = (\theta(k) - \theta(-k))\frac{1}{\sqrt{2\pi m}}\sqrt{|k|}e^{i\frac{t_A k^2}{2m}} \quad . \tag{4.115}$$

The symmetric states are

$$g^s_{t_A}(k) = (\theta(k) + \theta(-k))\frac{1}{\sqrt{2\pi m}}\sqrt{|k|}e^{i\frac{t_A k^2}{2m}} \tag{4.116}$$

Ho ever, the operator is not self-adjoint and these states are not orthogonal.

$$\langle t'_A|t_A\rangle = \frac{1}{2\pi m}\int_0^\infty dk^2\, e^{\frac{i}{2m}k^2(t_A - t'_A)} \quad = \delta(t_A - t'_A) -$$

Trying to make $\mathbf{T}$ self-adjoint by defining boundary conditions at $k=0$ leads to the requirement on square integrable ave functions $u(k)$,$v(k)$ such that

$$\langle u, \mathbf{T}v\rangle - \langle \mathbf{T}^{*}u, v\rangle = i\,m\left[\lim_{k\to 0^-}\frac{v(k)\overline{u(k)}}{|k|} + \lim_{k\to 0^+}\frac{v(k)\overline{u(k)}}{|k|}\right] = 0 \tag{4.118}$$

i.e.. the boundary conditions must be chosen so that $\frac{v(k)\overline{u(k)}}{k}$ is continuous through $k=0$. This continuity condition cannot force $u(k)$ to have the same boundary conditions as $v(k)$ for any choice of boundary condition on $v(k)$. For example, if e choose $v(k)/\sqrt{k}$ to be continuous through the origin, then $u(k)/\sqrt{k}$ must be anti-continuous through the origin. I.e. the domain of definition of $\mathbf{T}$ and $\mathbf{T}^{*}$ differ and $\mathbf{T}$ cannot be self-adjoint. This is not at all surprising, given the proof in Section 4.2.

One might ho ever, try to modify $\mathbf{T}$ in order to make it self-adjoint in the manner sho n in Section 4.3 [y y m ε ε − k am]. Consider the operator

$$\mathbf{T}_{\epsilon}(k) = -im\sqrt{f_{\epsilon}(k)}\frac{d}{dk}\sqrt{f_{\epsilon}(k)}$$

$f_{\epsilon}(k)$ is some smooth function hich differs from $1/k$ only near $k=0$. Since d $v(k)$ could diverge at the origin at a rate approaching $1/\sqrt{k}$ and still remain square-integrable, if $f_{\epsilon}(k)$ goes to zero at least as fast as k, then $\mathbf{T}_{\epsilon}$ ill be self-adjoint and defined over all square integrable functions. Ho ever, as e sho in Sections 4.5 and 4.6, these eigenstates do not behave as one ould expect a time of arrival eigenstate to behave.

It can be verified that $\mathbf{T}_{\epsilon}$ has a degenerate set of eigenstates $|t_A,+\rangle$ for $k>$ $t_A,-\rangle$ for $k<0$, given by

$$g^{\pm}_{t_A}(k) = \langle k|t_A,\pm\rangle = \theta(\pm k)\frac{1}{\sqrt{2\pi m}}\frac{1}{\sqrt{f_{\epsilon}(k)}}e^{\frac{it_A}{m}\int_{\pm\epsilon}^{k}\frac{1}{f_{\epsilon}(k}}$$

states given by

$$f_\epsilon(k) = \begin{cases} \frac{k}{\epsilon^2} & |k| < \epsilon \\ \frac{1}{k} & |k| > \epsilon \end{cases} \tag{4.121}$$

If $\epsilon \to 0$, one might expect $\mathbf{T}_\epsilon$

Its modulus squared vanishes hen integrated around a small neighborhood of $x = 0$. ${}_\epsilon\tilde{g}^+(x)_{t_A}$ then, is not localized around the point of arrival, at the time-of-arrival. This ill also be verified in the next section here e examine normalizable states. Although ${}_\epsilon\tilde{g}^+(x)_{t_A}$ is not localized around the point of arrival at the time of arrival, one might hope that this part of the state does not contribute significantly in time-of-arrival measurements hen $\epsilon \rightarrow$

As argued in the previous section, the second term should act like a time-of-arrival state. The first term is due to the modification of **T** and has nothing to do ith the time of arrival. We ill first sho that the second term can indeed be localized at the point-of-arrival $x = 0$ at the time of arrival $t = t_A$. We ill do this by expanding it around $x = 0$ in a Taylor series. After taking the limit $\epsilon \rightarrow 1$ 1 ,U here

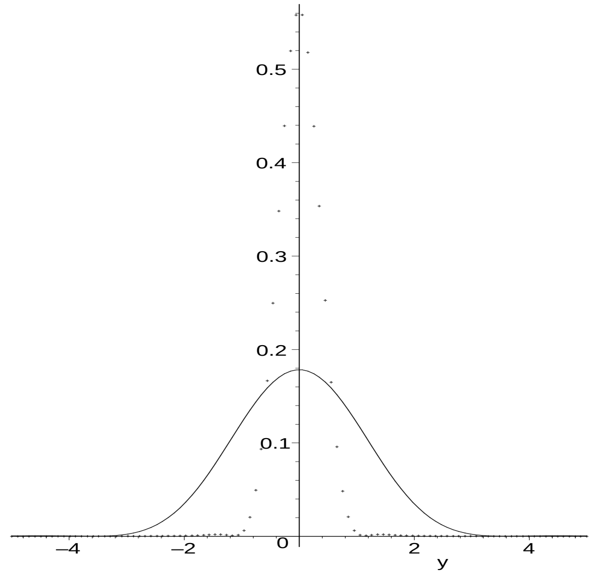


Figure 4.1: Unmodified part of time-of-arrival eigenstate. $|_o\tau^+(x,\tau)|^2$ vs. x, ith $\Delta = m$ (solid line), and $\Delta = \frac{m}{10}$ (dashed line). As Δ gets smaller, the probability function gets more and more peaked around the origin.

If $i\epsilon x$ is not large, e can use the fact that for Δ and ϵ very small, $i\epsilon^2 t_A/m \ll 1/2$ so that e have

$$ {}_\epsilon\tau^+(x,0) \simeq (2\pi)^{\frac{1}{4}}\sqrt{\frac{\epsilon^3\Delta}{2m}}\frac{\Phi(\sqrt{-i\epsilon x})}{\sqrt{-i\epsilon x}} $$

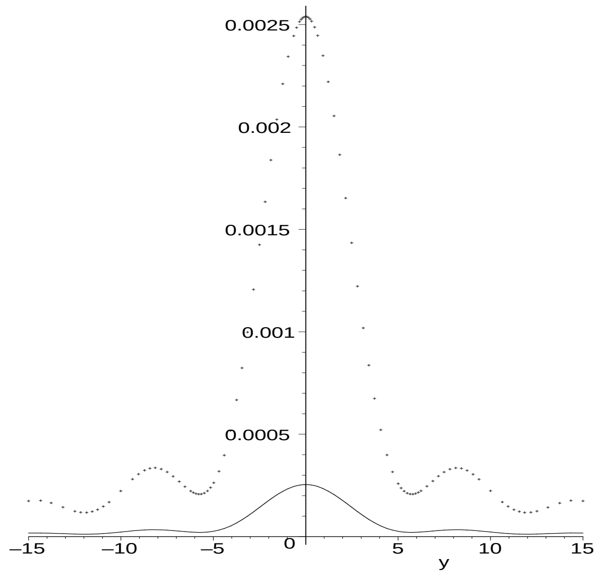


Figure 4.2: Modified part of time-of-arrival eigenstate. $\frac{1}{\epsilon}|{}_{\epsilon}\tau^{+}(x,\tau)|^2$ vs. ϵx, of $\sqrt{\frac{m}{\Delta}}x$. ith $\Delta\epsilon^2 = \frac{m}{10}$ (solid line) and $\Delta\epsilon^2 = \frac{m}{100}$ (dashed line). As Δ or ϵ gets smaller, the probability function drops near the origin, and gro s longer tails hich are exponentially far a s vs.Tr . ro s lo

As ϵ or Δ go to zero, N_ϵ diverges, and if e renormalize the state, the entire norm ill be made up of the modified part of the eigenstate.

It is also of interest to calculate the average value of the kinetic energy for these states, in order to see hether these states ill trigger the physical clocks discussed in Chapter 3. In calculating the average energy, the modified piece ill not matter since k^2 goes to zero at $k = 0$ faster than $\frac{1}{\sqrt{k}}$ diverges. We find

$$
\begin{aligned}
\langle\tau_\Delta^+|\mathbf{H}_k|\tau_\Delta^+\rangle &= \int dk \frac{k^2}{2m}\langle\tau_\Delta^+|k\rangle\langle k|\tau_\Delta^+\rangle \\
&= \frac{N^2}{\pi(2m)^2}\int_0^\infty k^3 e^{\frac{i(t_A-t'_A)k^2}{2m}} e^{-\frac{t_A^2+t_A'^2}{\Delta^2}} dt_A\, dt'_A\, dk \\
&= \frac{4}{\Delta\sqrt{2\pi}}
\end{aligned}
\tag{4.142}
$$

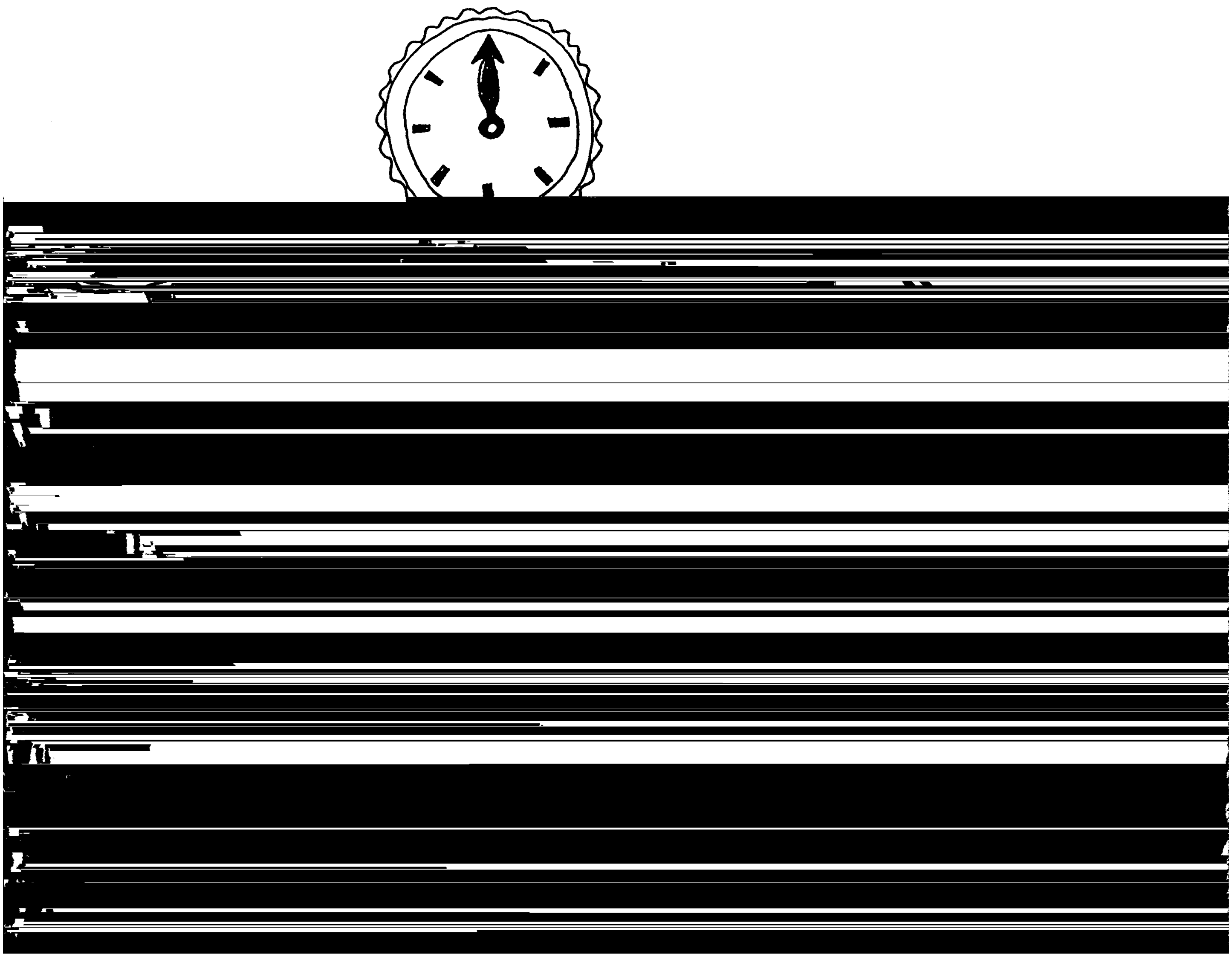
We see therefore, that the kinematic spread in arrival times of these states is proportional to $1/\bar{E}_k$. Since the probability of triggering the model clocks discussed in Chapter 3 decays as $\sqrt{E_k \delta t_A}$, here δt_A is the accuracy of the clock, e find that the states $|\tau_\Delta^+\rangle$ ill not al ays trigger a clock hose accuracy is $\delta t_A = \Delta$.

4.7 Limited Physical Meaning of Time-of-Arrival Operators

We ould also like to stress that continuous measurements differ both conceptually and quantitatively from a measurement of the time-of-arrival operator. Operationally one performs here t o completely different measurements. While the time-of-arrival operator is a formally constructed operator hich can be measured by an impulsive von-Neumann interaction, it seems that continuous measurements are much more closer to actual experimental set-ups. Furthermore, e have seen that the result of these t o measurements do not need to agree, in particular in the high accuracy limit, continuous measurements give rise to entirely different behavior. This suggests that as in the case of the problem of finding a "time operator" [20] for closed quantum systems, the time-of-arrival operator has a some hat limited physical meaning.

Chapter 5

Traversal Time



5.1 A Limitation on Traversal Time Measurements

5.2 Measuring Momentum Through Traversal-Distance

The measurement of traversal-distance may be considered the space-time *dual* of the measurement of traversal time: instead of fixing x_1 and x_2 and measuring $t_F = t_2 - t_1$, one fixes t_1 and t_2 and measures $x_F = x_2 - x_1$. It is instructive to examine first this simpler case of traversal-distance and point out the similarities and the differences.

Unlike the case of traversal time, a measurement of traversal-distance can be described by the standard von Neumann interaction. For a free particle the Hamiltonian is

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \mathbf{Q}\mathbf{x}\Big[\delta(t - t_1) - \delta(t - t_2)\Big] \tag{5.144}$$

here $\mathbf{Q}$ is the coordinate conjugate to the pointer variable $\mathbf{P}$. The change in $\mathbf{P}$ yields the traversal-distance:

$$\mathbf{P}(t > t_2) - \mathbf{P}_0 = \mathbf{x}(t_2) - \mathbf{x}(t_1) = \mathbf{x}_F. \tag{5.145}$$

Ho ever the measurement of the traversal-distance provides additional information: it also determines the momentum $\mathbf{p}$ of the particle *during* the time interval $t_1 < t < t_2$. From the equations of motion ar T

It is also commonly accepted that the d ell time operator [31], given by

$$\tau_{\mathbf{D}} = \int_0^\infty dt \mathbf{\Pi}_{x_A}(t) \tag{5.150}$$

here

$$\mathbf{\Pi}_{x_A}(0) = \int_{x_1}^{x_2} |x\rangle\langle x| \tag{5.151}$$

can be used to compute the traversal time[1]. Such a quantity ho ever, cannot be measured, since, as e sa in Chapter 2, the operator $\mathbf{\Pi}_{x_A}(t)$ does not commute ith itself at different times.

$$[\mathbf{\Pi}_{x_A}(t), \mathbf{\Pi}_{x_A}(t')] \neq 0. \tag{5.152}$$

Therefore, one must measure the traversal time in a more physical ay. One must demand that if e measure the traversal time to be t_F, then the particle must actually traverse the distance bet een x_1 and x_2 in the time given by the traversal time measurement. For example, one could have a clock hich runs hen the particle is bet een x_1 and x_2 given by the Hamiltonian [16][32]

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x})\mathbf{Q} \tag{5.153}$$

here the traversal time is given by the variable $\mathbf{P}$ conjugate to $\mathbf{Q}$ and the potential V is equal to 1 hen $x_1 \leq x \leq x_2$ and 0 every here else . In the Heisenberg picture, the equations of motion are

$$\dot{\mathbf{x}} = \mathbf{p}/m, \quad \dot{\mathbf{p}} = -\mathbf{Q}(\delta(\mathbf{x} - x_1) - \delta(\mathbf{x} - x_2)) \tag{5.154}$$

$$\dot{\mathbf{P}} = V(\mathbf{x}), \quad \dot{\mathbf{Q}} = 0. \tag{5.155}$$

The particle's momentum is disturbed during the measurement

$$\mathbf{p}' = \sqrt{\mathbf{p}^2 - 2m\mathbf{Q}} \tag{5.156}$$

[1] in our case, where there is no potential barrier, the dwell time and traversal time are equivalent

here $\mathbf{p}'$

here the particle has gone through the detector $P = t_F = \frac{mL}{p_o}$. We therefore need the condition

$$dP < \frac{mL}{p_o}. \tag{5.162}$$

Since at best e have $dP = 1/dQ$, e find

$$dp' dx = dp' L > 1. \tag{5.163}$$

The uncertainty relation (5.158) only applies to this particular model clock - it might be possible to accurately measure the traversal time in some clever ay. In the follo ing section e ill argue that this cannot be done, by demonstrating that this uncertainty applies to all measurements of traversal time.

Finally, e should note that a traversal time detector could be made by measuring the time-of-arrival to x_1 and the time-of-arrival to x_2. This ould require t o time-of-arrival clocks each ith its o n inaccuracy, hereas the model above only has one clock.

5.4 General Argument for a Minimum Inaccuracy

We no consider general measurements of traversal time. We ill ho ever, impose some fairly unrestrictive requirements on the measurement. We ill assume that the measurement does not prevent the particle from actually traversing the distance bet een x_1 and x_2. I.e. e ant to be able to say that the particle did indeed traverse the distance L - other ise, it is unclear hat it is that e are measuring.

We also demand that the result of the measurement corresponds in some sense ith the classical notion of traversal time. I.e., e are measuring something like mL/p. Our measuring device ill consist of a pointer, hich is set to some initial position P_o ith an uncertainty in the initial value of the pointer of dP. At the end of the measurement, e assume that the value of the traversal time is inferred accurately by reading the final value of the pointer. The measured traversal time is then proportional to $P_f - P_i$. The relative accuracy of the traversal time ill then be given by $\delta T_f / T_f - dP/(P_f - P_i)$

Another condition e ill impose is that the inaccuracy of the measurement, δT_F, is less than the quantity e are trying to measure $\mathbf{T}_F$ (i.e. e are looking at accurate measurements). Finally, e assume that the experimentalist has no kno ledge of the state of the particle, and thus must set the initial state of the measuring device (and its inaccuracy δP) ith no prior kno ledge of the ensemble.

Before proceeding ith the argument, e should be clear to distinguish bet een different types of uncertainties. For an operator $\mathbf{A}$, there exists a kinematic uncertainty hich e ill denote by $d\mathbf{A}$ given by

$$d\mathbf{A}^2 = \langle \mathbf{A}^2 \rangle - \langle \mathbf{A} \rangle^2. \tag{5.164}$$

This is the uncertainty in the distribution of the observable A. There is also the inherent *inaccuracy* in the measuring device. This is the relevant quantity in equations (5.143) and (5.158). It refers to the uncertainty in the initial state of the measuring device's pointer position P, and e ill denote it by δA. For our measuring devices e have

$$\delta A = dP_o \tag{5.165}$$

This inaccuracy applies to each individual measurement. Lastly, there is the uncertainty ΔA hich applies to the spread in measurements made on the ensemble. Given a set A_M of experiments $i = 1, 2, 3...$ hich yield results A_i, e have

$$\Delta A^2 =< A_M^2 > - < A_M >^2 . \tag{5.166}$$

This uncertainty includes a component due to the kinematic uncertainty of the attribute of the system, and also the inaccuracy of the device. For our measuring device, the kinematic spread in the pointer position at the end of each experiment gives ΔA

$$\Delta A = dP_f \tag{5.167}$$

The Heisenberg uncertainty relationship $dAdB > 1$ applies to measurements on ensembles. Given an ensemble, e measure **A** on half the ensemble and **B** on the other half. The uncertainty relation also applies to simultaneous measurements [2]. If e measure **A** and **B** simultaneously on each system in the ensemble, then the distributions of **A** and **B** must still satisfy the uncertainty relationship.

Returning no to the traversal time, e see that it can be interpreted as a simultaneous measurement of position and momentum. We kno the particle's momentum p during the time that it as bet een $x = x_1$ and $x = x_2$ from the classical equations of motion

$$t_F = \frac{mL}{p}. \tag{5.168}$$

In other ords, eigenstates of momentum must have traversal times given by equation (5.168). This measurement of momentum is analogous to the measurement described in the particle and so a measurement of $\mathbf{X}$ is also a measurement of $\mathbf{x}$. This is hat e mean by a local interaction.

and does not depend on the nature of the ensemble upon hich e ill be making measurements. For a free Hamiltonian, a measurement of the traversal time ill result in a final pointer position given by

$$P_f = P_o + \frac{mL}{p} \tag{5.172}$$

here p is the momentum of the particle in the absence of any measuring device. For eigenstates of $\mathbf{p}$ (or states peaked highly in p), e demand that the traversal time be given by the classically expected value [3]. Recall that the kinematic spread in the particle's momentum is given by $d\mathbf{p}^2 = \langle \mathbf{p}^2 \rangle - \langle \mathbf{p} \rangle^2$. A measurement of the traversal time for a particular experiment i can take on the values

$$\begin{aligned} T_i &= P_f \\ &= \frac{mL}{p} + \epsilon \end{aligned} \tag{5.173}$$

A given measurement T_i ill allo us to infer the momentum of the particle p_i during the measurement

$$p_i(T_i) = \frac{mL}{T_i} = \frac{mLp}{mL + p\epsilon}. \tag{5.174}$$

The average value of any po er α of the measured momentum is

$$\langle p_M \rangle = \int \left(\frac{mLp}{mL + p\epsilon} \right) \; f(p)g(\epsilon)dpd\epsilon \tag{5.175}$$

here $f(p)$ gives the distribution of the particle's momentum and $g(\epsilon)$ is the distribution of the fluctuations. We no choose the mass m of the ensemble so that e al ays have

$$\epsilon p \ll mL. \tag{5.176}$$

In other ords, e consider measurements on ensembles here the measurement is much more accurate than the quantity being measured. i.e.. $\delta T_F << T_F$. Indeed for the

[3] It is possible to include small deviations from the classical value, by including an additional term in (5.172). These fluctuations need to average to zero in order to satisfy the correspondence principle. For small fluctuations, the following discussion is not altered.

example given in the previous section, for every given ϵ and p

then implies

$$\delta T_F^2 > \frac{1 - \frac{1}{4}L^2 dp^2}{\langle E \rangle^2 + dE^2}. \tag{5.186}$$

No e note that e can arrange our experiment ith Ldp arbitrarily small, by choosing dp [illegible]

Chapter 6

Order of Events

[redacted]

6.1 Past and Future

The notion that events proceed in a ell defined sequence is unquestionable in classical

If the state evolves irreversible to a state for hich $\Pi_A \Psi(t) = 1$, then e can easily measure hether the event A has occurred at any time t. We could therefore measure hether a free particle arrives to a given location before or after a classical time t_B. Of course, for many systems, the system ill not irreversible evolve to the required state. For example, a particle influenced by a potential may cross over the origin many times. Ho ever, for an event such as atomic decay, the probability of the atom being re-excited is relatively small, and one can argue that the event is more or less irreversible.

For the case of a free particle hich is traveling to ards the origin from $x < 0$ one can argue that if at a later time I measure the projection operator onto the positive axis and find it there, then the particle must have arrived to the origin at some earlier time. This is in some sense a definition, because e kno of no ay to measure the particle being at the origin ithout altering its evolution or being extremely lucky and happening to measure the particle's location hen it is at the origin.

While measuring hether an event happened before or after a fixed time t_B may be possible, e ill find that for t o quantum systems, one cannot in general measure hether the time t_A of event A, occurred before or after the time t_B of event B.

In Section 6.2, confining ourselves to a particular example of order of events, e ill consider the question of order of arrival in quantum mechanics. Given t o particles, can b Œ 's l . ē, SM Œ, UM me I yp pa. l n ouM an eanythebabni measure mrli in r bn o ng the order of arrival by measuring the operator

$$\mathbf{O} = sgn(\mathbf{T}_x - \mathbf{T}_y) \tag{6.189}$$

here T_x and T_y are the time-of-arrival operators associated ith each particle.

In Section 6.3 e discuss measurements of coincidence. I.e., can e determine hether both particles arrived at the same time. Such measurements allo us to change the accuracy of the device before each experiment. We find that the measurement fails hen the accuracy is made better than $1/\bar{E}$.

In Section 6.4 e discuss the relationship bet een ordering of events and the resolving po er of Heisenberg's microscope, and argue that in general, one cannot prepare a t o particle state hich is al ays coincident to ithin a time of $1/\bar{E}$.

6.2 Which first?

We no examine a case here the time t_B is not given by a classical clock, but rather a quantum system. Consider t o free particles (hich e ill label as x and y) initially localized to the right of the origin, and traveling to the left. We then ask hether one can measure hich particle arrives to the origin first. The Hamiltonian for the system and measuring apparatus is given by

$$\mathbf{H} = \frac{\mathbf{P_x}}{2m} + \frac{\mathbf{P_y}}{2m} + \mathbf{H}_i \tag{6.190}$$

here $\mathbf{H}_i$ is some interaction Hamiltonian. For example, a promising interaction Hamiltonian is

$$\mathbf{H}_i = \alpha\delta(\mathbf{x})\theta(-\mathbf{y}) \tag{6.191}$$

ith α going to infinity. If the y-particle arrives before the x-particle, then the x-particle ill be reflected back. If the y-particle arrives after the x-particle, then neither particle

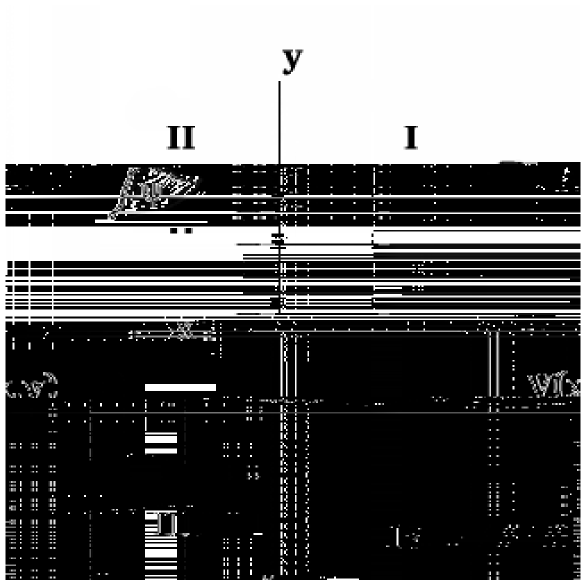


Figure 6.3: A potential hich can be used to measure hich of t o particles came first (given by $V(x, y) = \alpha\delta(\mathbf{x})\theta(-\mathbf{y})$). The ave function for t o incoming particles in one dimension looks like a single ave packet in t o dimensions travelling to ards the origin.

sees the potential, and both particles ill continue traveling past the origin. One can therefore ait a sufficiently long period of time, and measure here the t o particles are. If both the x and y particles are found past the origin, then e kno that the x-particle arrived first. If the y-particle is found past the origin hile the x-particle has been reflected back into the positive x-axis then e kno that the y-particle arrived first.

Classically, this method ould appear to unambiguously measure hich of the t o particle arrived first. Ho ever, in quantum mechanics, this method fails. From (6.190) e can see that the problem of measuring hich particle arrives first is equivalent to deciding here a single particle traveling in a plane arrives. T o particles localized to the right of the origin is equivalent to a single particle localized in the first quadrant (see Figure 6.3). The question of hich particle arrives first, becomes equivalent to the

to the origin (the sharp edge of the potential). The amplitude for being scattered off the region around the edge in the direction θ is given by $|f(r,\theta)|^2$.

It might be argued that since these particles scattered, they must have scattered off the potential, and therefore they represent experiments in hich the y-particle arrived first. Ho ever, this ould clearly over count the cases here the y-particle arrived first. We could have just as easily have placed our potential on the negative x-axis, in hich case, e ould over count the cases here the x-particle arrived first.

In the "interference region" e cannot have confidence that our measurement orked at all. We should therefore define a "failure cross section" given by

$$\begin{aligned} \sigma_f &= \int_0^{2\pi} |f(\theta)|^2 \\ &= \frac{1}{k\cos(\frac{\theta_o}{2})} \end{aligned} \tag{6.195}$$

From (6.195) e can see that cross section for scattering off the edge is the size of the particle's avelength multiplied by some angular dependence. Therefore, if the particle arrives ithin a distance of the origin given by

$$\delta x > 2/k \tag{6.196}$$

the measurement fails. We have dropped the angular dependence from (6.195) – the angular dependence is not of physical importance for measuring hich particle came first, as it depends on the details of the potential (boundary conditions) being used. The particular potential e have chosen is not symmetrical in x and y. From this e can

In other ords, our measurement procedure relies on making an inference bet een time measurements and spatial coordinates. The last t o equations then give us

$$\delta t > \frac{1}{E} \quad . \tag{6.198}$$

One ill not be able to determine hich particle arrived first, if they arrive ithin a time $1/E$ of each other, here E is the total kinetic energy of both particles. Note that Equation (6.198) is valid for a plane ave ith definite momentum k. For ave functions for hich $dk << k$, one can replace E by the expectation value $\langle E \rangle$. Ho ever, for ave functions hich have a large spread in momentum, or hich have a number of distinct peaks in k, then to ensure that the measurement almost al ays orks, one must measure the order of arrival ith an accuracy given by

$$\delta t > \frac{1}{\bar{E}} \tag{6.199}$$

here $\bar{E}$ is the minimum typical total energy [1].

Although it seemed plausible that one could measure hich particle arrived first, e found that if the particles are coincident to ithin $1/\bar{E}$, then the measurement fails.

6.3 Coincidence

In the previous model for measuring hich particle arrived first, e found that if the t o particles arrived to ithin $1/\bar{E}$ of each other, the measurement did not succeed. The idth $1/\bar{E}$ as an inherent inaccuracy hich could not be overcome. Ho ever, in our simple model, e ere not able to adjust the accuracy of the measurement.

It is therefore instructive to consider a measurement of "coincidence" alone for hich one can quite naturally adjust the accuracy of the experiment. Given t o particles

[1]For example, one need not be concerned with exponentially small tails in momentum space, since the contribution of this part of the wave function to the probability distribution will be small. If however, $\psi(E)$ has two large peaks at E_{small} and E_{big} spread far apart, then if δt does not satisfy $\delta t > 1/E_{small}$ one will get a distorted probability distribution. For a discussion of this, see Chapter 3.

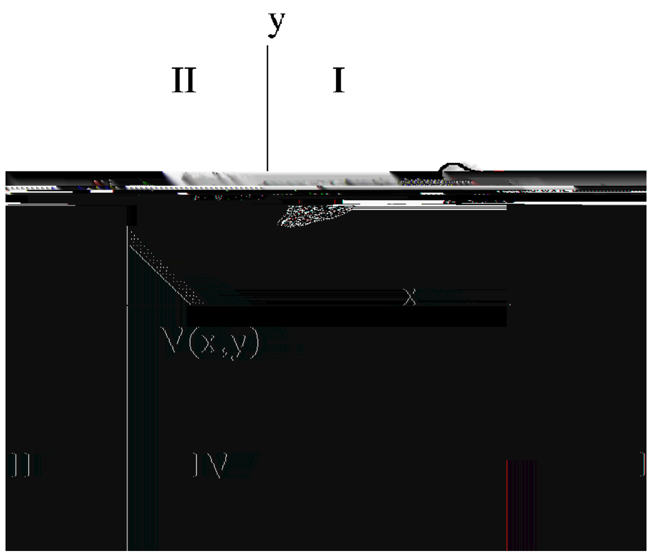


Figure 6.4: Potential for measuring hether t o particles are coincident.

traveling to ards the origin, e ask hether they arrive ithin a time δt_c of each other. If the particles do not arrive coincidently, then e do not concern ourselves ith hich arrived first. The parameter δt_c can be adjusted, depending on ho accurate e ant our coincident " mechanically, e once again find an interference region around the strip hich scatters particles into the classically forbidden regions of quadrant t o and four. The shado is not sharp, and e are not al ays certain hether the particles ere coincident.

A solution to plane aves scattering off a narro strip is ell kno n and can be found in many quantum mechanical texts (see for example [46] here the scattered ave is ritten as a sum of products of Hermite polynomials and Mathieu functions). Ho ever, for our purposes, e ill find it convenient to consider a simpler model for measuring coincidence, namely, an infinite circular potential of radius a, centered at the origin.

$$H_i = \alpha V(r/a) \tag{6.200}$$

here $V(x)$ is the unit disk, and e take the limit $\alpha \to \infty$.

It is ell kno n that if $a < 1/k$, then there ill not be a ell-defined shado behind the disk. To see this, consider a plane ave coming in from negative x-infinity. It can be expanded in terms of the Bessel function $J_m(kr)$ and then ritten asymptotically ($r \gg 1$) as a sum of incoming and outgoing circular aves.

$$\begin{aligned} e^{ikx} &= \sum_{m=0}^{\infty} \epsilon_m i^m J_m(kr) \cos m\theta \\ &\simeq \sqrt{\frac{1}{2\pi ikr}} \left[e^{ikr} \sum_{m=0}^{\infty} \epsilon_m \cos m\theta + ie^{-ikr} \sum_{m=0}^{\infty} \epsilon_m \cos m(\theta - \pi) \right] \quad . \end{aligned} \tag{6.201}$$

here ϵ_m is the Neumann factor hich is equal to 1 for $m = 0$ and equal to 2 other ise.

Since it can be sho n that

$$\sum_{m=0}^{M} \epsilon_m \cos m\theta = \frac{\sin\,(M + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta} \tag{6.202}$$

The t o infinite sums approach $2\pi\delta(\theta$ [illegible] a scattered ave

$$\psi = e^{ikx} + \frac{e^{ikr}}{\sqrt{r}} f(r\theta) \tag{6.203}$$

here

$$\frac{e^{ikr}}{\sqrt{r}} f(r,\theta) = -i \sum_{m=0}^{\infty} \epsilon_m e^{\frac{1}{2}m\pi i - i\delta_m} \sin\delta_m H_m(kr) \cos m\theta \quad , \tag{6.204}$$

$H_m(kr)$ are Hermite polynomials and

$$\tan\delta_m = \frac{-J_m(ka)}{N_m(ka)} \tag{6.205}$$

($N_m(ka)$ are Bessel functions of the second kind). For large values of r, the ave function can be ritten in a manner similar to (6.201), except that the outgoing ave is modified by the phase shifts δ_m.

$$\psi \simeq \frac{1}{\sqrt{2\pi ik}} i \sum_{m=0}^{\infty} \epsilon_m \cos m(\theta-\pi) \frac{e^{-ikr}}{\sqrt{r}} + \frac{e^{ikr}}{\sqrt{r}} f(r,\theta) \quad . \tag{6.206}$$

here

$$f(r,\theta) \simeq \frac{1}{\sqrt{2\pi ik}} \sum_{m=0}^{\infty} \epsilon_m e^{-2i\delta_m(ka)} \cos m\theta \tag{6.207}$$

In the limit that $ka \gg m$ the phase shifts can be ritten as

$$\delta_m \simeq ka - \frac{\pi}{2}(m + \frac{1}{2}) \quad . \tag{6.208}$$

In the limit of extrememely large a (but $a < r$), the outgoing aves then behave as

$$f(r,\theta) \simeq \lim_{M\to\infty} -i \frac{1}{\sqrt{2\pi ik}} e^{-2ika} \frac{\sin\,(M+\frac{1}{2})(\theta-\pi)}{\sin\frac{1}{2}(\theta-\pi)} \tag{6.209}$$

here once again e see that the angular distribution goes as the delta function $\delta(\theta-\pi)$. The disk scatters the plane ave directly back, and a sharp shado is produced. We see therefore, that in the limit of $ka \gg 1$, our measurement of coincidence orks.

The differential cross section can in general be ritten as

$$\begin{aligned} \sigma &= |f(\theta)|^2 \\ &= |\sum_{m=0}^{\infty} \epsilon_m e^{-2i\delta_m(ka)} \cos m\theta|^2 \end{aligned} \tag{6.210}$$

For $ka \gg 1$ (but still finite), (6.210) can be computed using our expression for the phase shifts from (6.208)

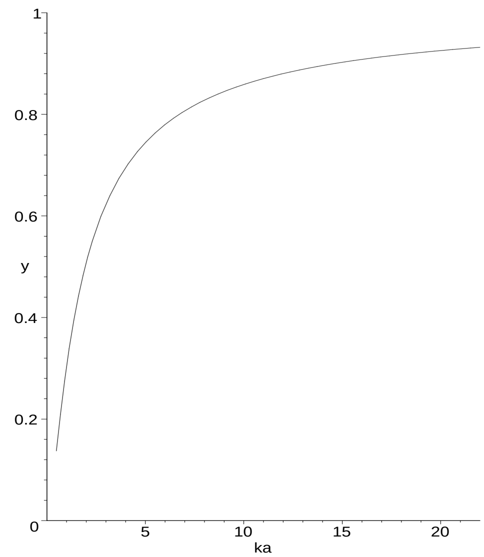


Figure 6.5: Phase shifts for coincidence detector ($\delta_1(ka)/\delta_0(ka)$ vs. ka)

depends on k. This is hat e require then, for the probability of our measurement to succeed independently of the energy of the incoming particles. From a plot of δ_1/δ_0 e see that this only occurs hen $ka \gg 1$ (Figure 6.5). Our condition for an accurate measurement is therefore that $a \gg 1/k$. Since $\delta t_c \simeq am/k$ $_c$ ≫ k fl

Preparing a state ψ_c corresponds to preparing a single particle in t o dimensions hich al ays arrives inside a region $\delta r = p\delta t_c/m$ of the origin. In other ords, suppose e ere to set up a detector of size δr at the origin. If a state ψ_c exists, then it ould al ays trigger the detector at some later time.

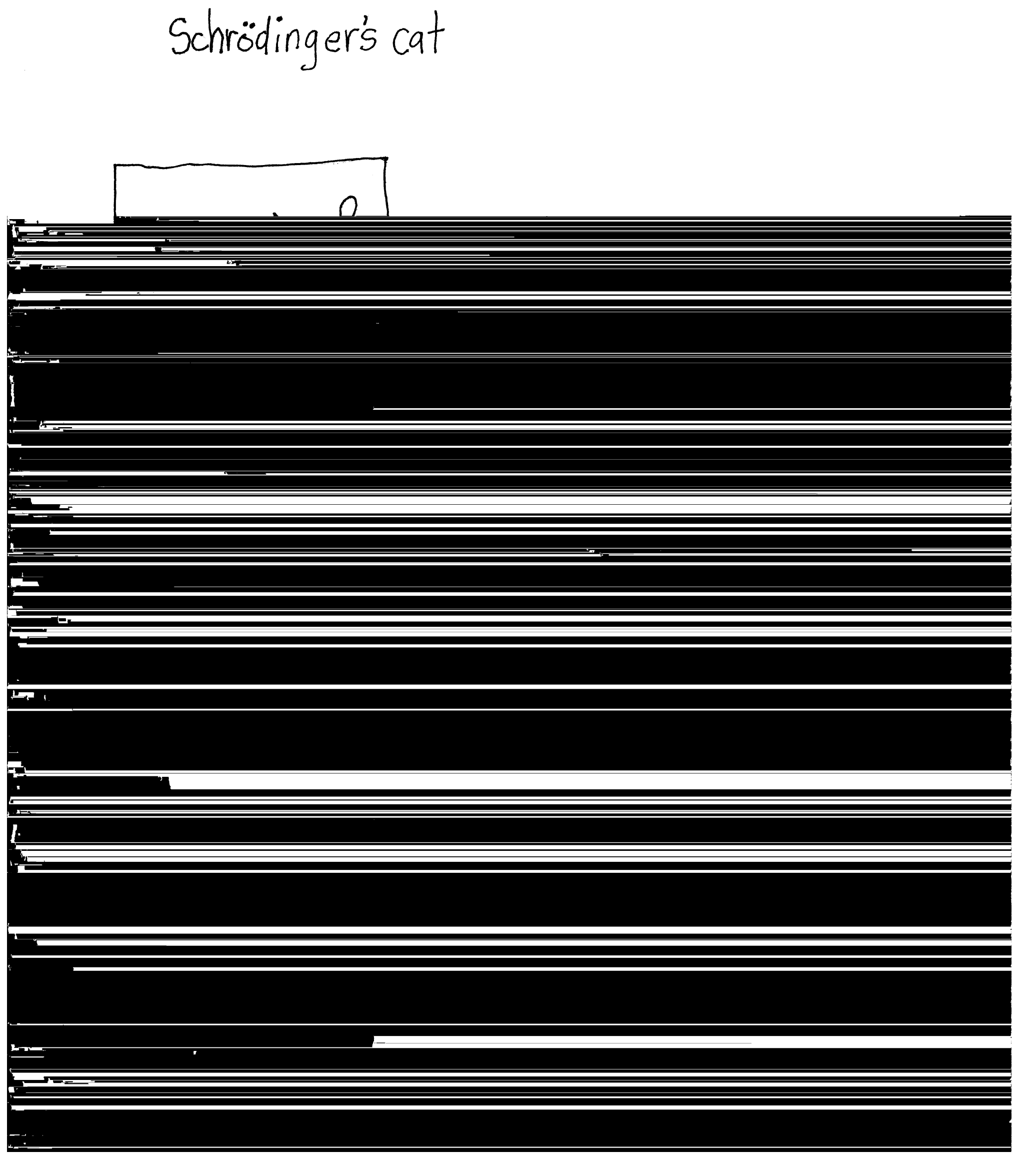
Our definition of coincidence requires that the state ψ_c

to do this to arbitrary accuracy, ithout affecting the system. In some sense, one may not be able to determine hich ay the light cone points.

One could use the arrival of arbitrarily energetic particles in order to denote space-time events, and although one can increase the energy of the particles in order to increase the accuracy ith hich one is able to measure the order of events, at some point the energy of the particles ill effect the curvature of the neighboring space time.

Chapter 7

Conclusion



We have argued that time plays a unique role in quantum mechanics. It is unlike other observables and one cannot naively assume it to be measurable. We have examined a number of different types of measurements of the time of an event, including measurements hich involve continual monitoring of the system, coupling to physical clocks, measuring of current operators, and time-of-arrival operators. These various types of measurements given different results, and there does not appear to be any canonical method for measuring the time of an event.

In the context of the time-of-arrival t_A, e have found a basic limitation on the accuracy (as opposed to uncertainty) that t_A can be determined reliably: $\delta t_A > 1/\bar{E}_k$. This limitation is quit different in origin from that due to the uncertainty principle; here it applies to the inference of the value of time for a *single* event. Furthermore, unlike the kinematic nature of the uncertainty principle, in our case the limitation is essentially dynamical in its origin; it arises hen the time-of-arrival is measured by means of a continuous interaction bet een the measuring device and the particle.

While e kno of no formal proof that this relation holds for time-of-arrival, our arguments are fairly general in nature. For the case of traversal time, e have argued that the limitation does not depend on any particular measurement procedure.

We have also argued that monitoring hether the particle is at the location of arrival x_A at various times, and also measuring the current operator, do not allo one to construct a probability distribution hich one could interpret as representing the probability that the particle ill arrive at a certain time.

We ould also like to stress that continuous measurements differ both conceptually and quantitatively from a measurement of the time-of-arrival operator. While the time-of-arrival operator is a formally constructed operator hich can be measured by an impulsive von-Neumann interaction, continuous measurements are much closer to actual experiments. Furthermore, e have seen that the result of these t o measurements do not need to agree. In particular, at high accuracy, continuous measurements give rise to entirely different behavior – the particle never arrives. The time-of-arrival on the other hand, can be measured to any accuracy.

Ho ever, the time-of arrival operator is not self-adjoint. Attempts to modify the time-of-arrival operator in such a ay as to make it self-adjoint result in the problem that the particle does not arrive on time ith probability $1/2$. Operators hich classically might give the time of an event cannot be given a physical interpretation. While several authors [9][25] have maintained that the problems ith defining an operator for the time of an event are technical, and can be circumvented by slightly modifying these operators. We have argued that probabilities in time are fundamentally different from traditional probabilities in quantum mechanics, and that there is a limitation on these measurements. As is the case ith "time operators" [20] in closed quantum systems, the time-of-arrival operator has a some hat limited physical meaning.

We have also seen that one cannot determine the temporal ordering of events to arbitrary accuracy. The limitation on these measurements is once again given by $1/\bar{E}$ here $\bar{E}$ is the typical total energy of the system. Nor can one prepare a t o particle system in a state in hich the t o particles al ays arrive ithin a time $1/\bar{E}$ of each other.

Ho ever as ith most research, this thesis raises more questions than it ans ers. Does a formalism exist here time is an element of reality? If not, does there exist a proof of the minimum inaccuracy bounds e have proposed? More intriguing, are some of the connections bet een this research, and the problem of time in quantum gravity and quantum cosmology imefly discussed th canonical approach to quantizing gravity. One immediately encounters th problemthat relative to th external parameter time in the Schrödinger equation, the state of the universe does not evolve. This is because the system must satisfy constraints hich are equivalent to reparametrization of the time variable.

The situation is some hat analogous to being inside a box, and having some external observer eigh the box ith high accuracy [40]. In order to keep the box at this fixed eight, the external experimenter cannot measure observables hich evolve in time. Quantum

These examples, hich arise out of trying to understand quantum gravity, have led us to examine the role that time plays in ordinary quantum

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Appendix A

Zero-Current Wavefunctions

One interesting aspect of the detector discussed in Section 3.3.2, is that hile it can be used for ave-packets arriving from the left or the right, it ill not al ays be triggered if the avefunction is a coherent superposition of right and left moving modes. Consider for example, the superposition

$$\psi(x) = Ae^{ikx} + Ae^{-ikx}. \tag{A.216}$$

One can easily verify that the current

$$j(x,t) = -i\frac{1}{2m}\left[\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial x} - \frac{\partial\psi^*(x,t)}{\partial x}\psi(x,t)\right] \tag{A.217}$$

is zero in this case. $|\psi(0,t)|^2$ is non-zero, although the state is not normalizable. As in eq. (3.63) this state evolves into

$$\langle x|\psi\rangle|\uparrow_z\rangle \quad \rightarrow \quad \frac{A}{\sqrt{2}}\Big[(e^{ikx} + e^{-ikx})|\uparrow_x\rangle + (e^{ikx} + e^{-ikx})|\downarrow_x\rangle\Big] \tag{A.218}$$

Which, hen re ritten in the σ_z basis, is just

$$A(e^{ikx} + e^{-ikx})|\uparrow_z\rangle. \tag{A.219}$$

i.e. the detector is never triggered.

This avefuntion is similar to the antisymmetric avefunctions discussed by Yamada and Takagi in the context of decoherent histories [36] and Leavens [37] in the context of Bohmian mechanics, here also one finds that the particles never arrive. Ho to best treat these cases is an interesting open question.

Appendix B

Gaussian Wave Packet and Clocks

Using the simple model of Section 3.3.1 (3.44), e no calculate the probability distribution of a clock hich measures the time-of-arrival of a Gaussian ave packet. We ill perform the calculation in the limits hen the clock is extremely accurate and extremely inaccurate. The ave function of the clock and particle is given by (3.52) and the distributions are both Gaussians given by (3.53). In the inaccurate limit, hen $p_o << k$, $A_{I'} \sim 1$. We trace over the position of the particle on the condition that the clock as triggered, ie. $x > 0$.

$$\begin{aligned} \rho(y,y)_{x>0} &= \int dx |\psi(x>0,y,t)|^2 \\ &\simeq N^2 \int_{-\infty}^{\infty} dk\, dk' \int_0^{\infty} dp\, dp'\, dx g(k) g^*(k') f(p) f^*(p') e^{i(q-q')x + i(p-p')y - \frac{i(q^2-q'^2)t}{2m}} \end{aligned} \tag{B.220}$$

After a sufficiently long time, ie. $t >> t_a$ the ave function has no support on the negative x-axis, and if $p_o > 1/\Delta y$, then it ill not have support in negative p. We can thus integrate p and x over the entire axis. Integrating over x gives a delta-function in q. We can then integrate over p' to give

$$\rho(y,y)_{x>0} \simeq \frac{2\pi N^2}{m} \int dk\, dk'\, dp \sqrt{k^2 + 2mp}\, g(k) g^*(k') f(p) f^*(p + \frac{k^2 - k'^2}{2m}) e^{i(k'^2 - k^2)\frac{y}{2m}}$$

here e have used the fact that $\delta(f(z)) = \frac{\delta(z-z_o)}{f'(z=z_o)}$ hen $f(z_o) = 0$. The square root term varies little in comparison ith the exponential terms and can be replaced by its average value $\sqrt{k_o^2 + 2mp_o} \simeq k_o$. Integrating over p gives

$$\rho(y,y)_{x>0} \simeq \frac{2\pi N^2 k_o}{m} \sqrt{\frac{\pi}{2\Delta y^2}} \int dk\, dk'\, e^{\frac{-\Delta y^2}{8m^2}(k+k')^2(k-k')^2} g(k) g^*(k') e^{i(k'^2-k^2)\frac{y}{2m}}. \tag{B.221}$$

Since $\Delta y\, k >> 1$, for a ave packet peaked around k_o e can approximate the argument of the first exponential by $\frac{-\Delta y^2 k_o^2}{2m^2}(k-k')^2$. This allo s us to integrate over k and k'

$$\rho(y,y)_{>0} \simeq \frac{1}{\sqrt{2\pi\gamma(y)}} e^{-\frac{(y-t_c)^2}{2\gamma(y)}} \tag{B.222}$$

here the idth is $\gamma(y) = \Delta y^2 + (\frac{m\Delta x}{k_o})^2 + (\frac{y}{2k_o\Delta x})^2$.

As expected, the distribution is centered around the classical time-of-arrival $t_c = x_o m/k_o$. The spread in y has a term due to the initial idth Δy in clock position y. The second and third term in $\gamma(y)$ is due to the kinematic spread in the time-of-arrival $1/dE = \frac{m}{kdk}$ and is given by $\frac{dx(y)\,m}{k_o}$ here $dx(y)^2 = \Delta x^2 + (\frac{y}{2m\Delta x})^2$. The y dependence in the idth in x arises because the ave packet is spreading as time increases, so that at later y, the ave packet is ider. As a result, the distribution differs slightly from a Gaussian although this effect is suppressed for particles ith larger mass.

When the clock is extremely accurate ie. $p_o >> k_o$ e have $A_{I'} \sim k\sqrt{\frac{2}{mp}}$.

$$\begin{aligned}
\rho(y,y)_{x>0} &\simeq \frac{2N^2}{m}\int_{-\infty}^{\infty} dk\, dk' \int_0^{\infty} dp\, dp'\, dx \frac{kk'}{\sqrt{pp'}} g(k) g^*(k') f(p) f^*(p') e^{i(q-q')x + i(p-p')y - \frac{i(q^2-q'^2)t}{2m}} \\
&\simeq \frac{4\pi N^2}{m}\int dk\, dk'\, dp \frac{kk'}{m}\sqrt{\frac{k^2+2mp}{p(p+\frac{k^2-k}{}}}
\end{aligned}$$

[illegible]

here the idth $\tilde{\gamma}(y) = \Delta x^2 + (\frac{y}{\quad}$

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Appendix C

Time-of-Arrival Eigenstates

We ill no sho that the eigenstates of [9]