

5.2 Measuring Momentum Through Traversal-Distance

The measurement of traversal-distance may be considered the space-time *dual* of the measurement of traversal time: instead of fixing x_1 and x_2 and measuring $t_F = t_2 - t_1$, one fixes t_1 and t_2 and measures $x_F = x_2 - x_1$. It is instructive to examine first this simpler case of traversal-distance and point out the similarities and the differences.

Unlike the case of traversal time, a measurement of traversal-distance can be described by the standard von Neumann interaction. For a free particle the Hamiltonian is

$$\mathbf{H} = \mathbf{p}^2$$

uncertainty relations, because \mathbf{p} commutes with \mathbf{x}_F , and \mathbf{x} remains completely uncertain. Similarly, in the case of the traversal time we shall see that the measurement determines also the momentum during the traversal, however unlike the present case, since the particle has to be in the interval $x_2 - x_1$ during the traversal, it is also a measurement of the location. This indicates that, in the latter case, in order not to violate the kinematic uncertainty principle for x and p , the accuracy with which the traversal time, T_F , or the momentum may be measured must be limited.

5.3 Measuring Traversal Time

For traversal time the classical equations of motion suggest that a traversal time operator might be given by

$$\mathbf{T}_F = \frac{mL}{\mathbf{p}}, \quad (5.149)$$

here $L = x_2 - x_1$. This operator is self-adjoint, but like the time-of-arrival operator, we shall see that different outcomes are found in a direct measurement of T_F and a measurement of the operator \mathbf{T}_F . Furthermore, one can measure the quantity \mathbf{T}_F at any time, so there is no reason to believe that the particle actually traveled between the two points in the time t_F . Since \mathbf{T}_F is only a function of

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here \mathbf{p}' is the particle's momentum during the measurement, and \mathbf{p} is the undisturbed momentum. However if the interaction is weak $Q \ll E_p$, then after a sufficient time, the clock will read the undisturbed traversal time

$$\begin{aligned} \mathbf{P}(t \rightarrow \infty) - \mathbf{P}(0) &\simeq \int_0^\infty V\left(\mathbf{x}(0) - \frac{\mathbf{p}_0 t}{m}\right) dt \\ &= \frac{m(x_2 - x_1)}{\mathbf{p}} \end{aligned} \quad (5.157)$$

If we require an accurate measurement of the traversal time, then a small dP will result in large values of the coupling Q . If Q is too large, the clock can reflect the particle at x_1 and one will obtain a traversal time equal to 0. This therefore imposes a restriction on the accuracy with which one can measure the traversal time. As in Chapter 3 we find that

$$\delta T_F > 1/E_p \quad (5.158)$$

is required in order to be able to measure the traversal time, and

$$\delta T_F \gg 1/E_p \quad (5.159)$$

in order to measure the undisturbed value of the traversal time.

Let us show that the above conditions are consistent with the uncertainty relations for the position and momentum. If (5.159) is satisfied, we have $Q \ll E$, and by eq. (5.156) the momentum during the measurement is

$$\mathbf{p}' \simeq \mathbf{p} - \frac{m}{\mathbf{p}} \mathbf{Q}. \quad (5.160)$$

Thus during the measurement, the momentum will be uncertain by an amount

$$dp' \simeq \frac{m}{p_o} dQ. \quad (5.161)$$

In order to know whether the particle entered our detector, we need to be able to distinguish between the case where the pointer is at its initial position $P = 0$, and the case

here the particle has gone through the detector $P = t_F = \frac{mL}{\hbar k}$

value of the pointer. The measured traversal time is then proportional to $P_f - P_i$. The relative accuracy of the traversal time will then be given by $\delta T_f/T_f - dP/(P_f - P_i)$

Another condition we will impose is that the inaccuracy of the measurement, δT_F , is less than the quantity we are trying to measure T_F (i.e. we are looking at accurate measurements). Finally, we assume that the experimentalist has no knowledge of the state of the particle, and thus must set the initial state of the measuring device (and its inaccuracy δP) with no prior knowledge of the ensemble.

Before proceeding with the argument, we should be clear to distinguish between different types of uncertainties. For an operator \mathbf{A} , there exists a kinematic uncertainty which we will denote by $d\mathbf{A}$ given by

$$d\mathbf{A}^2 = \langle \mathbf{A}^2 \rangle - \langle \mathbf{A} \rangle^2. \quad (5.164)$$

This is the uncertainty in the distribution of the observable A . There is also the inherent *inaccuracy* in the measuring device. This is the relevant quantity in equations (5.14 i

$$\delta T_F \ll \Delta$$

The Heisenberg uncertainty relationship $dAdB > 1$ applies to measurements on ensembles. Given an ensemble, we measure **A** on half the ensemble and **B** on the other half. The uncertainty relation also applies to simultaneous measurements². If we measure **A** and **B** simultaneously on each system in the ensemble, then the distributions of **A** and **B** must still satisfy the uncertainty relationship.

Returning now to the traversal time, we see that it can be interpreted as a simultaneous measurement of position and momentum. We know the particle's momentum p during the time that it is between $x = x_1$ and $x = x_2$ from the classical equations of motion

$$t_F = \frac{mL}{p}. \quad (5.168)$$

In other words, eigenstates of momentum must have traversal times given by equation (5.168). This measurement of momentum is analogous to the measurement described in Section 5.2. Instead of measuring the change in position at two specified times t_1 and t_2 , we are now measuring the difference in arrival times after specifying two different positions x_1 and x_2 . During the measurement, we also know that the particle is somewhere between $x = x_1$, and $x = x_2$. i.e.. we know that $x = \frac{x_1+x_2}{2} \pm L/2$.

The uncertainty relationship also applies to these measured quantities $\Delta x \Delta p > 1$. This essentially means that a detector of size L will disturb the momentum by at least $2/L$, so that repeated measurements on an ensemble will give $\Delta p > 2/L$. The position of the detector **X** commutes with the momentum of the particle **p** [10] however, we demand that the particle actually travel the distance L . The particle must actually be inside the detector during the measurement. As a result, **X** must be coupled to the position **x** of

²For a discussion of how the uncertainty relation applies to simultaneous measurements, see for example, Arthurs and Kelly[38] They propose a model for simultaneous measurements using a Hamiltonian $\mathbf{H} = \delta(t)(\mathbf{P}_1\mathbf{A} + \mathbf{P}_2\mathbf{B})$ which measures the variables **A** and **B** using two measurement pointers **Q**₁ and **Q**₂ which are conjugate to **P**₁ and **P**₂. They show how the Heisenberg uncertainty relation applies to the uncertainty in the outcomes of the measurement of **A** and **B**.

the particle and so a measurement of \mathbf{X} is also a measurement of \mathbf{x} . This is what we mean by a local interaction.

We now show why we expect (5.143) to be true. During the measurement of traversal

and does not depend on the nature of the ensemble upon which we will be making measurements. For a free Hamiltonian, a measurement of the traversal time will result in a final pointer position given by

$$P_f = P_o + \frac{mL}{p} \quad (5.172)$$

here p is the momentum of the particle in the absence of any measuring device. For eigenstates of \mathbf{p} (or states peaked highly in p), we demand that the traversal time be given by the classically expected value³. Recall that the kinematic spread in the particle's momentum is given by $d\mathbf{p}^2 = \langle \mathbf{p}^2 \rangle - \langle \mathbf{p} \rangle^2$. A measurement of the traversal time for a particular experiment i can take on the values

$$\begin{aligned} T_i &= P_f \\ &= \frac{mL}{p} + \epsilon \end{aligned} \quad (5.173)$$

A given measurement T_i will allow us to infer the momentum of the particle p_i during the measurement

$$p_i(T_i) = \frac{mL}{T_i} = \frac{mLp}{mL + p\epsilon}. \quad (5.174)$$

The average value of any pointer α of the measured momentum is

$$\langle p_M \rangle = \int \left(\frac{mLp}{mL + p\epsilon} \right) f(p)g(\epsilon)dpd\epsilon \quad (5.175)$$

here $f(p)$ gives the distribution of the particle's momentum and $g(\epsilon)$ is the distribution of the fluctuations. We now choose the mass m of the ensemble so that we always have

$$\epsilon p \ll mL. \quad (5.176)$$

In other words, we consider measurements on ensembles where the measurement is much more accurate than the quantity being measured. i.e.. $\delta T_F \ll T_F$. Indeed for the

³It is possible to include small deviations from the classical value, by including an additional term in (5.172). These fluctuations need to average to zero in order to satisfy the correspondence principle. For small fluctuations, the following discussion is not altered.

example given in the previous section, for every given

then implies

$$\delta T_F^2 > \frac{1 - \frac{1}{4}L^2 dp^2}{\langle E \rangle^2 + dE^2}. \quad (5.186)$$

Note that we can arrange our experiment with Ldp arbitrarily small, by choosing dp of the ensemble arbitrarily small. i.e.. the uncertainty in the traversal time is small. As a result, in order to ensure that Heisenberg's uncertainty relation is never violated, we must have

$$\delta T_F > \frac{1}{\sqrt{\langle E \rangle^2 + dE^2}}. \quad (5.187)$$

Since dp is small, we can write

$$\delta T_F > \frac{1}{\langle E \rangle}. \quad (5.188)$$

It is interesting to note that since the momentum operator commutes with the free Hamiltonian, the restriction on traversal time measurements only comes from the dynamical considerations given above.

5.5 From Traversal Time to Barrier Tunneling Time

We have seen that the measurement of the traversal time given two positions cannot be made arbitrarily accurate. We have argued this by looking at a simple model for measuring traversal time, and we have also given a model independent, qualitative proof of this which applies when the measurement only disturbs the system slightly. Finally we have given a more rigorous argument which applies when the uncertainty in traversal time is small. This strongly suggests that the limitation on measurements of arrival times is a general rule and not just an artifact of the types of models considered so far. Operators for both the traversal time and the arrival time don't seem to correspond to physical (continuous) processes. The case of traversal time is different from time-of-arrival in that there does exist a self-adjoint traversal time operator, and the semi-bounded spectrum of the Hamiltonian does not seem to play an important role in the

restriction on measurement accuracy. The accuracy restriction on traversal time may be particularly important for experiments on barrier tunneling time. If one uses a physical clock to measure the time it takes for a particle to travel from one location to another, with a barrier situated somewhere between the two locations [17][32], then the accuracy of this clock may affect the tunneling particle. The limitation presented in this Chapter seem to indicate that measurements of barrier tunneling times could also need to be inherently inaccurate, because if one tries to measure the tunneling time too accurately, the particle may be unable to tunnel. Our result concerning traversal time indicates that the barrier tunneling time also cannot be precisely defined.