4.1 Indirect Time-of-Arrival Measurements

In the previous chapter, e sa that one cannot measure the time-of-arrival of a free particle to arbitrary accuracy by coupling the particle to a clock. Still, one can imagine an indirect determination of arrival time by a measurement of some regularized time-of-arrival operator $\mathbf{T}(\mathbf{x}(t), \mathbf{p}(t), x_A)$ [9]. In quantum mechanics, ordinary observables like position and momentum *are* represented by operators at a fixed time t. Ho ever, e ill sho that there is no operator associated ith the time it takes for a particle to arrive to a fixed location. In Section 4.2 e ill prove formally that in general a Hermitian time-of-arrival operator ith a continuous spectrum can only exist for systems ith an unbounded Hamiltonian. This is because the existence of a time-of-arrival operator requires the existence of a time operator hich is conjugate to the Hamiltonian. As is argued in Section 4.3, since $\mathbf{T}$ can be measured ith arbitrary accuracy it does not correspond to the result obtained by the direct measurement discussed in Chapter 3 .

In Section 4.4 e sho hy the time-of-arrival operator for a free particle is not self-adjoint, and explore the possible modifications that can be made in order to make it self-adjoint. The idea is that by modifying the operator in a very small neighborhood around $k = 0$, one can formally construct a modified time-of-arrival operator hich behaves in much the same ay as the unmodified time-of-arrival operator.

We then explore some of the properties of the modified time-of-arrival states. In Section 4.5 e examine normalizable states hich are coherent superpositions of time-of-arrival eigenstates, and discuss the possibility of localizing these states at the location of arrival at the time-of-arrival. Our results for the "unmodified" part of the time-of-arrival state seem to agree ith those of Muga, Leavens and Palao ho have studied these states independently [30]. In Section 4.6 e sho that in an eigenstate of the modified time-of-arrival operator, the particle, at the predicted time-of-arrival, is found far a ay from the point of arrival ith probability 1/2. We also calculate the average energy of the states, in order to relate them to our proposal in Chapter 3 that one cannot measure the time-of-arrival to an accuracy better than $1/\bar{E}_k$. We end ith concluding remarks in Section 4.7.

4.2 Conditions on A Time-of-Arrival Operator

As discussed in the previous section, although a direct measurement of the time-of-arrival may not be possible, one can still try to observe it indirectly by measuring some operator $\mathbf{T}(\mathbf{p}, \mathbf{x}, x_A)$. In the next t o sections e shall examine this operator and its relation to the continuous measurements described in the previous chapters. First in this section e sho that an exact time-of-arrival operator cannot exist for systems ith bounded Hamiltonian.

To begin ith, let us start ith the assumption that the time-of-arrival is described, as other observables in quantum mechanics, by a Hermitian operator $\mathbf{T}$.

$$\mathbf{T}(t)|t_A\rangle_t = t_A|t_A\rangle_t \tag{4.95}$$

Here the subscript $\rangle_t$ denotes the time dependence of the eigenkets, and $\mathbf{T}$ may depend explicitly on time. Hence for example, the probability distribution for the time-of-arrival for the state

$$|\psi\rangle = \int g(t'_A)|t'_A\rangle dt'_A \tag{4.96}$$

ill be given by $prob(t_A) = |g(t_A)|^2$. We shall no also assume that the spectrum of $\mathbf{T}$ is continuous and unbounded: $-\infty < t_A < \infty$.

Should $\mathbf{T}$ correspond to time-of-arrival it must satisfy the follo ing obvious condition. $\mathbf{T}$ must be a constant of motion and in the Heisenberg representation

$$\frac{d\mathbf{T}}{dt} = \frac{\partial \mathbf{T}}{\partial t} + \frac{1}{i}[\mathbf{T}, H] = 0. \tag{4.97}$$

That is, the time-of-arrival cannot change in time. If, for example, I measure that the bus is supposed to arrive at 7 p.m., then if I make another measurement at some other time, I should still find that the bus should (or did) arrive at 7 p.m

For a time-independent Hamiltonian, time translation invariance implies that the eigenkets $|t_A\rangle_t$ depends only on $t - t_A$, i.e. the eigenkets cannot depend on the absolute time t. This means for example that at the time of arrival: $|t_A\rangle_{t=t_A} = |t'_A\rangle_{t=t'_A}$. Time-translation invariance implies

$$|t_A\rangle_t = e^{-i\mathbf{G}}|0\rangle_0. \tag{4.98}$$

here $\mathbf{G} = \mathbf{G}(t - t_A)$ is a hermitian operator. Therefore, $|t_A\rangle_t$ satisfies the differential equations

$$i\frac{\partial}{\partial t_A}|t_A\rangle_t = \frac{\partial \mathbf{G}}{\partial t_A}|t_A\rangle_t = -\frac{\partial \mathbf{G}}{\partial t}|t_A\rangle_t, \qquad i\frac{\partial}{\partial t}|t_A\rangle_t = \frac{\partial \mathbf{G}}{\partial t}|t_A\rangle_t. \tag{4.99}$$

No act on the eigenstate equation (4.95) ith the differential operators $i\partial_{t_A}$ and $i\partial_t$. This yields

$$-\mathbf{T}\frac{\partial \mathbf{G}}{\partial t}|t_A\rangle_t = -t_A\frac{\partial \mathbf{G}}{\partial t}|t_A\rangle_t + i|t_A\rangle_t, \tag{4.100}$$

and

$$i\frac{\partial \mathbf{T}}{\partial t}|t_A\rangle_t + \mathbf{T}\frac{\partial \mathbf{G}}{\partial t}|t_A\rangle_t = t_A\frac{\partial \mathbf{G}}{\partial t}|t_A\rangle_t. \tag{4.101}$$

By adding the t o equations above, the dependence on

conjugate to H. The value of $\mathbf{T}'$ is recorded on the conjugate of q – call it P_q. No the uncertainty is given by $dT'_A = d(P_q$

state of the particle if the clock has stopped, and $|\chi'(t_A)\rangle$ the final state of the particle if the clock has not stopped.

Since the eigenstates of **T** form a complete set, e can express any state of the particle as $|\psi\rangle = \int dt_A C(t_A)|t_A\rangle$. We then obtain :

$$\int dt_A C(t_A)|t_A\rangle|y=t_0\rangle \;\rightarrow\; \int dt_A C(t_A)|\chi(t_A)\rangle|y=t_A\rangle + \left(\int dt_A C(t_A)|\chi'(t_A)\rangle\right)|y=t\rangle. \tag{4.112}$$

The final probability to measure the time-of-arrival is hence $\int dt_a |C(t_a)\chi(t_a)|^2$. On the other hand e found that for a general ave function ψ, in the limit of $dt_a \rightarrow 0$, the probability for detection vanishes. Since the states of the clock, $|y=t_a\rangle$, are orthogonal in this limit, this implies that $\chi(t_a)=0$ in eq. (4.111) for all t_A. Therefore, the eigenstates of **T** cannot trigger the clock.

It should be mentioned ho ever, that one ay of circumventing this difficulty may be to consider a coherent set of **T** eigenstates states instead of the eigenstates themselves. These normalizable states ill no longer be orthogonal to each, so they may trigger the clock if they have sufficient energy [1]. In this regard it is of interest to prematurely quote a result hich e ill sho in Section 4.5 - the average energy of a Gaussian distribution of time-of- arrival eigenstates is proportional to $1/\Delta$ here Δ is the spread of the Gaussian. This puts us at the edge of the limitation given in Equation (3.91).

---

[1] An arbitrary wave packet can be written as a superposition of normalized eigenstates, and yet we know that arbitrary wave packets do not trigger the clocks of Section 3.3.1. This creates a somewhat interesting situation if normalized eigenstates trigger a clock, but wave packets made of superpositions of them do not.

4.4 The Modified Time-of-Arrival Operator

Kinematically, one expects that the time-of-arrival operator for a free particle arriving at the location $x_A = 0$ might be given by

$$\mathbf{T} = -\frac{m}{2}\frac{1}{\sqrt{\mathbf{p}}}\mathbf{x}(0)\frac{1}{\sqrt{\mathbf{p}}}. \tag{4.113}$$

The operator $-m(\frac{1}{\mathbf{p}}\mathbf{x} + \mathbf{x}\frac{1}{\mathbf{p}})$ is equivelant to the one above as can be seen by use of the commutation relations for $\mathbf{x}$ and $\mathbf{p}$.

In the k representation this operator can be ritten as

$$\mathbf{T}(k) = -im\frac{1}{\sqrt{k}}\frac{d}{dk}\frac{1}{\sqrt{k}} \quad = -im(\frac{1}{k}\frac{d}{dk} + \frac{d}{dk}\frac{1}{k}) \tag{4.114}$$

 here $\sqrt{k} = i\sqrt{|k|}$ for $k < 0$. If one solves the eigenvalue equation, one finds a set of anti-symmetric states [2] for this operator given by

$$g^a_{t_A}(k) = (\theta(k) - \theta(-k))\,\frac{1}{\sqrt{2\pi m}}\sqrt{|k|}e^{i\frac{t_A k^2}{2m}} \quad . \tag{4.115}$$

The symmetric states are

$$g^s_{t_A}(k) = (\theta(k) + \theta(-k))\,\frac{1}{\sqrt{2\pi m}}\sqrt{|k|}e^{i\frac{t_A k^2}{2m}} \tag{4.116}$$

Ho ever, the operator is not self-adjoint and these states are not orthogonal.

$\langle$

Trying to make $\mathbf{T}$ self-adjoint by defining boundary conditions at $k=0$ leads to the requirement on square integrable ave functions $u(k)$,$v(k)$ such that

$$\langle u, \mathbf{T}v\rangle - \langle \mathbf{T}^{*}u, v\rangle = i\,m\left[\lim_{k\to 0^-}\frac{v(k)\overline{u(k)}}{|k|} + \lim_{k\to 0^+}\frac{v(k)\overline{u(k)}}{|k|}\right] = 0 \qquad (4.118)$$

i.e.. the boundary conditions must be chosen so that $\frac{v(k)\overline{u(k)}}{k}$ is continuous through $k=0$. This continuity condition cannot force $u(k)$ to have the same boundary conditions as $v(k)$ for any choice of boundary condition on $v(k)$. For example, if e choose $v(k)/\sqrt{k}$ to be continuous through the origin, then $u(k)/\sqrt{k}$ must be anti-continuous through the origin. I.e. the domain of definition of $\mathbf{T}$ and $\mathbf{T}^{*}$ differ and $\mathbf{T}$ cannot be self-adjoint. This is not at all surprising, given the proof in Section 4.2.

One might ho ever, try to modify $\mathbf{T}$ in order to make it self-adjoint in the manner sho n in Section 4 states given by

$$f_\epsilon(k) = \begin{cases} \frac{k}{\epsilon^2} & |k| < \epsilon \\ \frac{1}{k} & |k| > \epsilon \end{cases} \tag{4.121}$$

If $\epsilon \to 0$, one might expect $\mathbf{T}_\epsilon$ to be a good approximation to the time of arrival operator hen acting on states that do not have support around $k = 0$ [9].

As e sho in Appendix C, hen these states are examined in the x-representation, and if one only considers the contribution to the Fourier transform of $g^+_{t_A}(k)$ from $|k| > \epsilon$ (i.e.. the "unmodified" part of the eigenstate), then one finds that at the time-of-arrival, the states are not delta functions $\delta(x)$ but are proportional to $x^{-3/2}$; they have support over all x. Ho ever, although the state has long tails out to infinity, the quantity $\int dx' |x'^{-3/2}|^2 \sim x^{-2}$ goes to zero as $x \to \infty$. Furthermore, the modulus squared of the eigenstates diverges hen integrated around the point of arrival $x = 0$. As a result, one might expect that the normalized state ill be localized at the point-of-arrival at the time-of-arrival. In Section 4.5 e sho that this is indeed so. Ho ever, the full eigenstate, is made up both of this "unmodified" piece, and a modified piece. The modified part of the eigenstate is not ell localized at the time-of-arrival. The contribution to the Fourier-transform of the state $g^+_{t_A}(k)$ from $0 < k < \epsilon$ is given by

$${}_\epsilon\tilde{g}^+(x)_{t_A} = \frac{\epsilon}{\sqrt{2\pi m}} \int_0^\epsilon \frac{dk}{\sqrt{k}} e^{ikx} e^{-it_A \frac{k^2}{2m}} e^{\frac{i\epsilon^2 t_A}{m} \ln \frac{k}{\epsilon}} \; . \tag{4.122}$$

Because $\mathbf{T}_\epsilon$ is no longer the generator of energy translations for $|k| < \epsilon$, $g^+_{t_A}(k)$ is not time-translation invariant. For the $t_A = 0$ state, (4.4) can be integrated to give

$${}_\epsilon\tilde{g}^+(x)_{t_A} = \frac{\epsilon}{\sqrt{2xim}} \Phi(\sqrt{i\epsilon x}) \tag{4.123}$$

here Φ is the probability integral. For large x, ${}_\epsilon\tilde{g}^+(x)_{t_A}$ goes as $\frac{1}{\sqrt{x}}$ and the quantity $\int dx' |{}_\epsilon\tilde{g}^+_{t_A}(x')|^2 \sim \ln x$ diverges as $x \to \infty$. For small x, ${}_\epsilon\tilde{g}^+_{t_A}(x)$ is proportional to $e^{-i\epsilon x}$.

Its modulus squared vanishes hen integrated around a small neighborhood of $x = 0$. ${}_\epsilon\tilde{g}^+(x)_{t_A}$ then, is not localized around the point of arrival, at the time-of-arrival. This ill also be verified in the next section here e examine normalizable states. Although ${}_\epsilon\tilde{g}^+(x)_{t_A}$ is not localized around the point of arrival at the time of arrival, one might hope that this part of the state does not contribute significantly in time-of-arrival measurements hen $\epsilon \to 0$. Ho ever, e ill no see that for coherent superpositions of these eigenstates, half the norm is made up of the modified piece of the eigenstate.

4.5 Normalized Time-of-Arrival States

Since the time-of-arrival states are not normalizable, e ill examine the properties of states $|\tau_\Delta\rangle$ hich are narro superpositions of the modified time-of-arrival eigenstates. These states are normalizable, although they are no longer orthogonal to each other [3] .

We can no consider coherent superpositions of these eigenstates

$$|\tau_\Delta^\pm\rangle = N\int dt_A |t_A,\pm\rangle e^{-\frac{(t_A-\tau)^2}{\Delta^2}}. \tag{4.124}$$

 here N is a normalization constant and is given by $N = (\frac{2}{\pi\Delta^2})^{1/4}$. The spread dt_A in arrival times is of order Δ.

We no examine hat the state $\tau(x,t)^+ = \langle x|\tau_\Delta^+\rangle$ looks like at the point of arrival as a function of time. In hat follo s, e ill ork ith the state centered around $\tau = 0$ for simplicity. This ill not affect any of our conclusions. $\tau^+(x,t)$ is given by

$$\begin{aligned}
\tau^+(x,t) &= N\int \langle x|e^{\frac{-i\mathbf{p}^2 t}{2m}}|t_A,+\rangle e^{-\frac{t_A^2}{\Delta^2}} dt_A \\
&= N\int_0^\epsilon e^{-\frac{t_A^2}{\Delta^2}} e^{\frac{-ik^2}{2m}t} e^{ikx} g_{t_A}^+(k) dt_A\, dk \quad + N\int_\epsilon^\infty e^{-\frac{t_A^2}{\Delta^2}} e^{\frac{-ik^2}{2m}t} e^{ikx} g_{t_A}^+(k) dt_A\, dk \\
&\equiv {}_\epsilon\tau^+(x,t) \; + {}_o\tau^+(x,t)
\end{aligned} \tag{4.125}$$

[3] These coherent states form a positive operator valued measure (POVM). The measurement of time-of-arrival using POVMs has been discussed in [26].

here

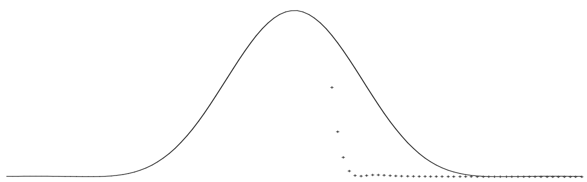
$$b_n = i^n 2^{n-\frac{3}{4}} \pi^{-\frac{1}{4}} \Gamma(\frac{3}{8} + \frac{n}{4}) \tag{4.131}$$

We see then that ${}_o\tau^+(x,0)$ is a function of $\sqrt{\frac{m}{\Delta}}x$ (ith a constant of $(\frac{m}{\Delta})^{1/4}$ out front). As a result, the probability of finding the particle in a neighborhood δ of x is given by

$$\int_{-\delta}^{\delta} |{}_o\tau^+(\sqrt{\frac{m}{\Delta}}x, 0)|^2 dx = \sqrt{\frac{\Delta}{m}} \int_{-\delta\sqrt{\frac{m}{\Delta}}}^{\delta\sqrt{\frac{m}{\Delta}}} |{}_o\tau^+(u,0)|^2 du. \tag{4.132}$$

Since $|{}_o\tau^+(u,0)|^2$ is proportional to $\sqrt{\frac{m}{\Delta}}$, and is square integrable, e see that for any δ, one need only make Δ small enough, in order to localize the entire particle in the region of integration. ${}_o\tau^+(x,t)$ is localized in a neighborhood δ around the point-of-arrival at the time-of-arrival as $\Delta \to 0$. The state is localized in a region δ of order $\sqrt{\frac{\Delta}{m}}$. This is hat one ould expect from physical grounds, since e have

$$dx \sim$$



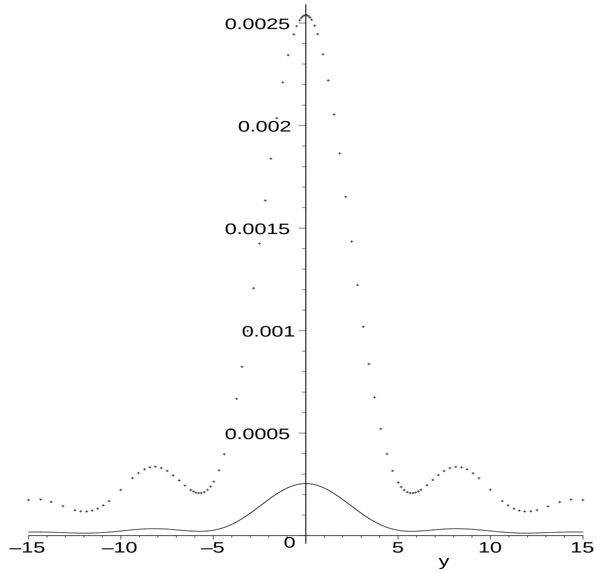


Figure 4.2: Modified part of time-of-arrival eigenstate. $\frac{1}{\epsilon}|_\epsilon\tau^+(x,\tau)|^2$ vs. ϵx, of $\sqrt{\frac{m}{\Delta}}x$. ith $\Delta\epsilon^2 = \frac{m}{10}$ (solid line) and $\Delta\epsilon^2 = \frac{m}{100}$ (dashed line). As Δ or ϵ gets smaller, the probability function drops near the origin, and gro s longer tails hich are exponentially far a ay.

$t = t_A = 0$ is sho n in Figure 4.2.

4.6 Contribution to the Norm due to Modification of T

We no sho that the modified part of $|\tau_\Delta^+\rangle$ contains at least half the norm, no matter ho small ϵ is made. The norm of the state $|\tau_\Delta^+\rangle$ can be ritten as

$$\begin{aligned}\int |\langle k|\tau_\Delta^+\rangle|^2 dk &= N^2\int_0^\epsilon |e^{-\frac{t_A^2}{\Delta^2}} g_{t_A}^+(k) dt_A|^2 dk \; + N^2\int_\epsilon^\infty |e^{-\frac{t_A^2}{\Delta^2}} g_{t_A}^+(k) dt_A|^2 dk \\ &\equiv N_\epsilon^2 + N_o^2 \end{aligned} \qquad (4.136)$$

here N_ϵ^2 is the norm of the modified part of the time-of-arrival state, and N_o^2 is the norm of the unmodified part. The second term can be inte

$$= \frac{1}{2} \tag{4.137}$$

here ithout loss of generality, e are looking at the state centered around $\tau = 0$ at $t = 0$.

The unmodified piece can contain only half the norm. The rest is found in the modified piece.

$$N_\epsilon^2 = \frac{N^2}{2\pi m}\int_0^\epsilon dk \int dt_A dt'_A$$

As ϵ or Δ go to zero, N_ϵ diverges, and if e renormalize the state, the entire norm ill be made up of the modified part of the eigenstate.

It is also of interest to calculate the average value of the kinetic energy for these states, in order to see hether these states ill trigger the physical clocks discussed in Chapter 3. In calculating the average energy, the modified piece ill not matter since k^2 goes to zero at $k = 0$ faster than $\frac{1}{\sqrt{k}}$ diverges. We find

$$\begin{aligned}
\langle \tau_\Delta^+ | \mathbf{H}_k | \tau_\Delta^+ \rangle &= \int dk \frac{k^2}{2m} \langle \tau_\Delta^+ | k \rangle \langle k | \tau_\Delta^+ \rangle \\
&= \frac{N^2}{\pi (2m)^2} \int_0^\infty k^3 e^{\frac{i(t_A - t'_A)k^2}{2m}} e^{-\frac{t_A^2 + t'^2_A}{\Delta^2}} dt_A \, dt'_A \, dk \\
&= \frac{4}{\Delta \sqrt{2\pi}}
\end{aligned} \tag{4.142}$$

We see therefore, that the kinematic spread in arrival times of these states is proportional to $1/\bar{E}_k$. Since the probability of triggering the model clocks discussed in Chapter 3 decays as $\sqrt{E_k \delta t_A}$, here δt_A is the accuracy of the clock, e find that the states $|\tau_\Delta^+\rangle$ ill not al ays trigger a clock hose accuracy is $\delta t_A = \Delta$.

4.7 Limited Physical Meaning of Time-of-Arrival Operators

We have seen that formally, a time-of-arrival operator cannot exist. If one modifies the time-of-arrival operator so as to make it self-adjoint, then its eigenstates no longer behave as one expects time-of-arrival states to behave. Half the time, a particle hich is in a time-of-arrival state ill not arrive at the predicted time-of-arrival. The modification also results in the fact that the states are no longer time-translation invariant.

For ave functions hich don't have support at $k = 0$, measurements can be carried out in such a ay that the modification ill not effect the results of the measurement. Nonetheless, after the measurement, the particle ill not arrive on time ith a probability of $1/2$. One cannot use $\mathbf{T}_\epsilon$ to prepare a system in a state hich arrives at a certain time.

We ould also like to stress that continuous measurements differ both conceptually and quantitatively from a measurement of the time-of-arrival operator. Operationally one performs here t o completely different measurements. While the time-of-arrival operator is a formally constructed operator hich can be measured by an impulsive von-Neumann interaction, it seems that continuous measurements are much more closer to actual experimental set-ups. Furthermore, e have seen that the result of these t o measurements do not need to agree, in particular in the high accuracy limit, continuous measurements give rise to entirely different behavior. This suggests that as in the case of the problem of finding a "time operator" [20] for closed quantum systems, the time-of-arrival operator has a some hat limited physical meaning.